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THE DYADIC OPERATOR APPROACH TO A STUDY IN CONICS, WITH SOME EXTENSIONS TO HIGHER DIMENSIONS

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1. Introduction. The discovery of a new truth in the older fields of mathematics is a rare event. Here an investigator may hope at best to secure greater elegance in method or notation, or to extend known results by some process of generalization.

It is our purpose, as the title indicates, to make a study of conic sections in the spirit of the above remark, using the symbolism developed by Gibbs.¹ Many of the facts we present were known to the Greek mathematicians, Menaechmus (c. 350 B.C.), Euclid (c. 300 B.C.), and Apollonius (c. 225 B.C.), and were developed further by the algebraic methods of Descartes (c. 1637 A.D.).² In fact our method of attack is suggested by Gibbs.³ We have found very meager use of it, however, in the literature available to us.

2. Assumed Relationships. In this section we state the preliminary assumptions, definitions, symbols, and theorems that we shall use later in the development of our topic.

We assume familiarity with the following: line-vectors, hereafter called vectors: \mathbf{a}, \mathbf{b} etc.; their magnitude: a, b , etc., respectively; their algebraic sum: $\mathbf{a} + \mathbf{b} = \mathbf{c}$; the scalar, or direct, product of two vectors: $\mathbf{a} \cdot \mathbf{b}$; non-coplanar vectors: $\mathbf{a}, \mathbf{b}, \mathbf{c}$; the mutually perpendicular unit vectors: $\mathbf{i}, \mathbf{j}, \mathbf{k}$; reciprocal vectors: $\mathbf{a}, \mathbf{a}^{-1}$; vectors $\mathbf{a}_1^{-1}, \mathbf{a}_2^{-1}, \mathbf{a}_3^{-1}$ called the reciprocal base system to non-coplanar base system $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$; a vector function; a vector equation; the position vector \mathbf{r} ; the derivative of a variable vector \mathbf{r} ; and the summation notation: $\mathbf{a} \cdot \mathbf{b}_i$ for $\sum \mathbf{a} \cdot \mathbf{b}_i$.

We shall let D stand for definition, T for theorem and P for an assumed property. For example, T a.b. will stand for the bth theorem of the ath section. T a.b.c. will be the cth corollary of T a.b. Also, we shall say \mathbf{r} when speaking of the point designated by the terminus of a position vector \mathbf{r} .

D 2.1. The symbol $\mathbf{a} \cdot \mathbf{b}_i$, $i = 1$ to n , denoted by a Greek letter, say Φ , is

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¹ Josiah Willard Gibbs (1839-1903) developed a vector algebra based on the fundamental ideas of Grassmann and Hamilton. The treatment of linear vector functions and the associated subject of dyadics constitute the most original part of his contribution to vector analysis; see A. P. Wills, *Vector and Tensor Analysis*, p. xxvi, xxvii.

² D. E. Smith, *History of Mathematics*, pp. 270-377.

³ J. W. Gibbs and E. B. Wilson, *Vector Analysis*, p. 272.

called a dyadic; if $n = 1$, a dyad. The vectors composing Φ shall be constant vectors.

D 2.2. The first vectors in each \mathbf{a}, \mathbf{b}_i are called the antecedents, the second, the consequents, of Φ .

D 2.3. \mathbf{b}, \mathbf{a}_i , denoted by Φ_c , is called the conjugate of the Φ of D 2.1.

D 2.4. $\Phi \cdot \mathbf{r}$ and $\mathbf{r} \cdot \Phi$ are defined to be $\mathbf{a}, \mathbf{b}_i \cdot \mathbf{r}$ and $\mathbf{r} \cdot \mathbf{a}, \mathbf{b}_i$, respectively. In the first, Φ is called a prefactor, in the second, a postfactor.⁴

D 2.5. $k(\mathbf{a}, \mathbf{b}_i)$, k a scalar, is defined to be equal $l_i(m, \mathbf{a}_i)(n, \mathbf{b}_i)s_i$ where the product $l_i m n s_i = k$ for every i .

P 2.1. The combination of vectors in a dyad is distributive;

$$\text{i.e.,} \quad (\mathbf{a} + \mathbf{b})\mathbf{a} = \mathbf{a}\mathbf{c} + \mathbf{b}\mathbf{c},$$

$$\text{and} \quad \mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{a}\mathbf{b} + \mathbf{a}\mathbf{c}.$$

$$\text{Likewise} \quad k_1\mathbf{a}\mathbf{b} + k_2\mathbf{a}\mathbf{b} = (k_1 + k_2)\mathbf{a}\mathbf{b}.$$

These conform to the laws of ordinary algebra, except that the order of the factors must be maintained.

P 2.2. The direct product of dyadics obeys the distributive law;

$$\text{i.e.,} \quad \Phi \cdot (\Psi + \Psi') = \Phi \cdot \Psi + \Phi \cdot \Psi',$$

$$(\Phi' + \Phi) \cdot \Psi = \Phi' \cdot \Psi + \Phi \cdot \Psi.$$

P 2.3. These laws hold for the direct product of dyadics and vectors when, and only when, the vectors occur at either or both ends of the list of factors of the product.

T 2.1. Any dyadic may be reduced to the sum of three dyads of which either the antecedents or the consequents, but not both, may be arbitrarily chosen non-coplanar vectors.⁵

D 2.6. If Φ can be reduced to the sum of three, two, or one dyad, it is said to be complete, planar, or linear, respectively. If, in the second case, the plane of the antecedents and the plane of the consequents coincide, Φ is said to be uniplanar. If, in the third case, the antecedent and the consequent are collinear, Φ is said to be unilinear.

The following are theorems on the operator Φ . We use notations $\mathbf{s} = \Phi \cdot \mathbf{r}$ and $\mathbf{t} = \mathbf{r} \cdot \Phi$.

T 2.2. If Φ is complete, \mathbf{s} and \mathbf{t} may be made to take on any desired values by suitable choice of \mathbf{r} .⁶

T 2.3. If Φ is planar, the vector \mathbf{s} may take on any value in the plane of the antecedents, and \mathbf{t} any value in the plane of the consequents of Φ , but no values out of those planes, by suitable choices of \mathbf{r} .⁷

T 2.4. If Φ is linear, the vector \mathbf{s} may take on any value collinear with the

⁴ Incidentally, $\Phi \cdot \mathbf{r}$ or $\mathbf{r} \cdot \Phi_c$, where \mathbf{r} is a variable vector, is a linear vector function of \mathbf{r} ; thus the operator $\Phi \cdot$ operating on \mathbf{r} accomplishes an affine transformation of space. We define $\Phi \cdot \Psi$, $\Phi = \mathbf{a}_i \mathbf{b}_i$, $\Psi = \mathbf{c}_j \mathbf{d}_j$ to be $\mathbf{a}_i(\mathbf{b}_i \cdot \mathbf{c}_j)\mathbf{d}_j = \Omega$.

⁵ Gibbs and Wilson, *op. cit.*, p. 271.

⁶ *Ibid.*, p. 285.

⁷ *Ibid.*, p. 285.

antecedents of Φ , and \mathbf{t} any value collinear with the consequents of Φ , but no other values.⁸

T 2.5. It is always possible to reduce a complete dyadic to the sum of three terms of which the antecedents among themselves and the consequents among themselves are mutually perpendicular, written as $\Phi = a\mathbf{i}'\mathbf{i} + b\mathbf{j}'\mathbf{j} + c\mathbf{k}'\mathbf{k}$. This is called the normal form of Φ .⁹ For uniplanar dyadics, the normal form of Φ is $a\mathbf{i}'\mathbf{i} + b\mathbf{j}'\mathbf{j}$.

D 2.7. If $\Phi \cdot \mathbf{r} = \Psi \cdot \mathbf{r}$, for every \mathbf{r} , $\Phi = \Psi$.

D 2.7.1. If $\mathbf{r} \cdot \Phi = \mathbf{r}$ and $\Phi \cdot \mathbf{r} = \mathbf{r}$, Φ shall be called an idemfactor, I , which is unique.

D 2.8. When
$$\Phi = a_{11}\mathbf{ii} + a_{12}\mathbf{ij} + a_{13}\mathbf{ik} + a_{12}\mathbf{jk} + a_{22}\mathbf{ji} + a_{23}\mathbf{jk} + a_{31}\mathbf{ki} + a_{32}\mathbf{kj} + a_{33}\mathbf{kk},$$

Φ is said to be in nonion form.

D 2.8.1. The idemfactor in nonion form is

$$I = \mathbf{ii} + \mathbf{jj} + \mathbf{kk}$$

D 2.9. If $\Phi = \mathbf{a}_i\mathbf{b}_i$ and $\Psi = \mathbf{b}_i^{-1}\mathbf{a}_i^{-1}$, then $\Psi = \Phi^{-1}$ is called the reciprocal of Φ and $\Phi^{-1} \cdot \Phi = I$.

D 2.10. If $\Phi = \Phi_c$, Φ is said to be self-conjugate and if $\Phi = -\Phi_c$, Φ is said to be anti-self-conjugate.

T 2.6. Any dyadic may be divided in one and only one way into the sum of two dyadics of which one is self-conjugate and the other anti-self-conjugate.¹⁰

T 2.7. Any self-conjugate dyadic may be expressed in the form, $\Phi = a\mathbf{ii} + b\mathbf{jj} + c\mathbf{kk}$ where a, b, c are scalars, positive or negative.¹¹

T 2.8. The equation $\mathbf{r} = \mathbf{a} \cos s + \mathbf{b} \sin s$ represents an ellipse with \mathbf{a} and \mathbf{b} as conjugate radii.¹²

T 2.9. As λ varies $\frac{x^2}{n^2 - \lambda} + \frac{y^2}{m^2 - \lambda} + \frac{z^2}{l^2 - \lambda} = 1$ represents the equation of a family of confocal central quadrics.¹³

T 2.10. If $\mathbf{r}' = \Phi \cdot \mathbf{r}$, $d\mathbf{r}' = \Phi \cdot d\mathbf{r}$; and if $w = \mathbf{u} \cdot \Phi \cdot \mathbf{v}$, $dw = d\mathbf{u} \cdot \Phi \cdot \mathbf{v} + \mathbf{u} \cdot \Phi \cdot d\mathbf{v}$.¹⁴

The two forms of dyadics used in the following discussion are known in the literature as tonic and special tonic.¹⁵

3. *The Parabola.* T 3.1. If Φ is unilinear, \mathbf{r} a variable vector, \mathbf{v} any constant vector, and k a constant scalar, then $\mathbf{r} \cdot \Phi \cdot \mathbf{v} = \mathbf{v} \cdot I \cdot \mathbf{r} + k$ represents two parallel lines or a parabola.

Let $\Phi = n\mathbf{jj}$, $\mathbf{v} = m\mathbf{j}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = m\mathbf{j} \cdot I \cdot \mathbf{r} + k$ gives $ny^2 = my + k$, two parallel lines.

⁸ *Ibid.*, p. 286.

⁹ *Ibid.*, p. 305.

¹⁰ *Ibid.*, p. 297.

¹¹ *Ibid.*, p. 305.

¹² Wills, *op. cit.*, p. 48.

¹³ D. M. Y. Sommerville, *Analytical Geometry of Three Dimensions*, p. 235.

¹⁴ Gibbs and Wilson, *op. cit.*, p. 404.

¹⁵ Wills, *op. cit.*, p. 154.

Take $\Phi = n\mathbf{j}\mathbf{j}$, $\mathbf{v} = a\mathbf{i} + b\mathbf{i}$; then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = (a\mathbf{i} + b\mathbf{i}) \cdot \mathbf{I} \cdot \mathbf{r} + k$ is equivalent to $ny^2 = ax + by + k$, a parabola.

We shall consider the parabola $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = \mathbf{i} \cdot \mathbf{I} \cdot \mathbf{r}$ where $\Phi = \frac{\mathbf{j}\mathbf{j}}{4a}$.

T 3.2. The equation of the polar line¹⁶ of the point \mathbf{a} is $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = \mathbf{i} \cdot \mathbf{I} \cdot \frac{(\mathbf{s} + \mathbf{a})}{2}$.

Let \mathbf{s} be a point in the polar line. A point on the line joining \mathbf{a} and \mathbf{s} is $\frac{y\mathbf{s} + x\mathbf{a}}{x + y}$. If this point is on the parabola, $\frac{y\mathbf{s} + x\mathbf{a}}{x + y} \cdot \Phi \cdot \frac{y\mathbf{s} + x\mathbf{a}}{x + y} = \mathbf{i} \cdot \mathbf{I} \cdot \frac{y\mathbf{s} + x\mathbf{a}}{x + y}$. This reduces to

$$(\mathbf{s} \cdot \Phi \cdot \mathbf{s} - \mathbf{i} \cdot \mathbf{I} \cdot \mathbf{s})(y/x)^2 + [2\mathbf{s} \cdot \Phi \cdot \mathbf{a} - \mathbf{i} \cdot \mathbf{I} \cdot (\mathbf{s} + \mathbf{a})]y/x + (\mathbf{a} \cdot \Phi \cdot \mathbf{a} - \mathbf{i} \cdot \mathbf{I} \cdot \mathbf{a}) = 0.$$

The two values of the ratio y/x determined by this equation are equal in magnitude and opposite in sign, since \mathbf{s} is on the polar line of \mathbf{a} . Consequently, the term in y/x vanishes. Thus $2\mathbf{s} \cdot \Phi \cdot \mathbf{a} - \mathbf{i} \cdot \mathbf{I} \cdot (\mathbf{s} + \mathbf{a}) = 0$ reduces to $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = \mathbf{i} \cdot \mathbf{I} \cdot \frac{\mathbf{s} + \mathbf{a}}{2}$, the required equation.

T 3.2.1. The reciprocal relationship, if \mathbf{b} is on the polar of \mathbf{a} , \mathbf{a} is on the polar of \mathbf{b} , is now readily established.

$$\text{The polar of } \mathbf{b} \text{ is } \mathbf{s} \cdot \Phi \cdot \mathbf{b} = \mathbf{i} \cdot \mathbf{I} \cdot \frac{(\mathbf{s} + \mathbf{b})}{2}. \quad (1)$$

$$\text{The polar of } \mathbf{a} \text{ is } \mathbf{s}' \cdot \Phi \cdot \mathbf{a} = \mathbf{i} \cdot \mathbf{I} \cdot \frac{(\mathbf{s}' + \mathbf{a})}{2}. \quad (2)$$

Since \mathbf{b} is on the polar of \mathbf{a} , (2) is satisfied by $\mathbf{s}' = \mathbf{b}$, or $\mathbf{b} \cdot \Phi \cdot \mathbf{a} = \mathbf{i} \cdot \mathbf{I} \cdot \frac{(\mathbf{b} + \mathbf{a})}{2}$, which is the same as (1) with \mathbf{a} substituted for \mathbf{s} .

T 3.2.2. The polar of a point \mathbf{r} on the parabola is the tangent to the curve at that point.

$$\mathbf{r} \cdot \Phi \cdot \mathbf{r} = \mathbf{i} \cdot \mathbf{I} \cdot \mathbf{r} \text{ has for its total differential } 2 \, d\mathbf{r} \cdot \Phi \cdot \mathbf{r} = \mathbf{i} \cdot \mathbf{I} \cdot d\mathbf{r}.$$

The total differential of $\mathbf{s} \cdot \Phi \cdot \mathbf{r} = \mathbf{i} \cdot \mathbf{I} \cdot \frac{(\mathbf{s} + \mathbf{r})}{2}$ is $d\mathbf{s} \cdot \Phi \cdot \mathbf{r} = \mathbf{i} \cdot \mathbf{I} \cdot \frac{d\mathbf{s}}{2}$, $2 \, d\mathbf{s} \cdot \Phi \cdot \mathbf{r} = \mathbf{i} \cdot \mathbf{I} \cdot d\mathbf{s}$; but this equation is satisfied by $d\mathbf{s} = d\mathbf{r}$, which proves the theorem.

T 3.4. $2 \, \Phi \cdot \mathbf{a} - \mathbf{i}$ is normal to the polar of \mathbf{a} . $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = \mathbf{i} \cdot \mathbf{I} \cdot \frac{(\mathbf{s} + \mathbf{a})}{2}$; $2 \, d\mathbf{s} \cdot \Phi \cdot \mathbf{a} = \mathbf{i} \cdot \mathbf{I} \cdot d\mathbf{s}$; $d\mathbf{s} \cdot (2 \, \Phi \cdot \mathbf{a} - \mathbf{i}) = 0$; which proves the theorem.

T 3.5. The locus of the mid-points of a system of parallel chords of a parabola is a straight line parallel to the axis of the parabola.

Let \mathbf{s} be the mid-point of one of the chords of the system parallel to \mathbf{a} . $\mathbf{r} = \mathbf{s} + x\mathbf{a}$ is the equation of the secant line upon which the chord lies. Solving for points of intersection of secant and parabola, we have, $(\mathbf{s} + x\mathbf{a}) \cdot \Phi \cdot (\mathbf{s} + x\mathbf{a}) = \mathbf{i} \cdot \mathbf{I} \cdot (\mathbf{s} + x\mathbf{a})$, which gives $x^2 \mathbf{a} \cdot \Phi \cdot \mathbf{a} + x(2 \, \mathbf{s} \cdot \Phi \cdot \mathbf{a} - \mathbf{i} \cdot \mathbf{I} \cdot \mathbf{a}) + \mathbf{s} \cdot \Phi \cdot \mathbf{s} - \mathbf{i} \cdot \mathbf{I} \cdot \mathbf{s} = 0$.

¹⁶ We assume the uniqueness of the harmonic division relationship of the four points determined by a line through the pole, intersecting the polar and the conic.

Since \mathbf{s} bisects the chosen chord the two solutions for x are equal in magnitude and opposite in sign. Then $2\mathbf{s} \cdot \Phi \cdot \mathbf{a} - \mathbf{i} \cdot \mathbf{a} = 0$.

Let $\mathbf{s} = n\mathbf{i} - m\mathbf{a}$; then $2m\mathbf{a} \cdot \Phi \cdot \mathbf{a} = \mathbf{i} \cdot \mathbf{a}$. $m = \frac{\mathbf{i} \cdot \mathbf{a}}{2\mathbf{a} \cdot \Phi \cdot \mathbf{a}} = \text{constant}$, which establishes the theorem.

One might continue establishing the various properties of the parabola in this way. Theorems, such as the following, are readily established.

T 3.6. The polar of the focus is the directrix.

T 3.7. A tangent at \mathbf{r} cuts the axis at $-\mathbf{r} \cdot \mathbf{i}$.

T 3.8. The tangents at the ends of a chord through the focus meet on the directrix.

$$\mathbf{s}' \cdot \Phi \cdot \mathbf{r} = \mathbf{i} \cdot \mathbf{i} \cdot \left(\frac{\mathbf{s}' + \mathbf{r}}{2} \right), \quad (1)$$

and

$$\mathbf{s}'' \cdot \Phi \cdot [\mathbf{a}\mathbf{i} - m(\mathbf{r} - \mathbf{a}\mathbf{i})] = \mathbf{i} \cdot \mathbf{i} \cdot \left[\frac{\mathbf{s}'' + \mathbf{a}\mathbf{i} + m(\mathbf{r} - \mathbf{a}\mathbf{i})}{2} \right] \quad (2)$$

are the equations of the required tangents.

If $\mathbf{s} = -\mathbf{a}\mathbf{i} + l\mathbf{j}$, l a variable scalar, satisfies these equations as values of \mathbf{s}' and \mathbf{s}'' , the theorem follows. By substitution (1) and (2) give $\frac{l\mathbf{j} \cdot \mathbf{r}}{4a} = \frac{-\mathbf{a} + \mathbf{i} \cdot \mathbf{r}}{2}$,

$$l = -2a \left(\frac{\mathbf{a} - \mathbf{i} \cdot \mathbf{r}}{\mathbf{j} \cdot \mathbf{r}} \right); \text{ and}$$

$$-\frac{l}{4a} (m\mathbf{j} \cdot \mathbf{r}) = \mathbf{i} \cdot \mathbf{i} \cdot [\mathbf{s} + \mathbf{a}\mathbf{i} - m(\mathbf{r} - \mathbf{a}\mathbf{i})], \quad l = -2a \left(\frac{\mathbf{a} - \mathbf{i} \cdot \mathbf{r}}{\mathbf{j} \cdot \mathbf{r}} \right),$$

which establishes the theorem.

$\mathbf{r} \cdot \Phi \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{i} \cdot \mathbf{r} + k = 0$ is only one type of an equation quadratic in the vector \mathbf{r} . The most general type of such equation would contain terms such as $\mathbf{r} \cdot \mathbf{r}$, $(\mathbf{r} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{r})$, $\mathbf{r} \cdot \mathbf{c}$ and $\mathbf{d} \cdot \mathbf{e}$, where \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} and \mathbf{e} are constant vectors. But $\mathbf{r} \cdot \mathbf{r} = \mathbf{r} \cdot \mathbf{i} \cdot \mathbf{r}$ and $(\mathbf{r} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{r}) = \mathbf{r} \cdot (\mathbf{a}\mathbf{b}) \cdot \mathbf{r}$. Then

$$\mathbf{r} \cdot \mathbf{r} + (\mathbf{r} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{r}) = \mathbf{r} \cdot (\mathbf{I} + \mathbf{a}\mathbf{b}) \cdot \mathbf{r} = \mathbf{r} \cdot \Phi \cdot \mathbf{r}.$$

But $\mathbf{r} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{i} \cdot \mathbf{r}$ and $\mathbf{d} \cdot \mathbf{e} = k$. Consequently,

$$\mathbf{r} \cdot \mathbf{r} + (\mathbf{r} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{r}) + \mathbf{r} \cdot \mathbf{c} + \mathbf{d} \cdot \mathbf{e} = \mathbf{r} \cdot \Phi \cdot \mathbf{r} + \mathbf{c} \cdot \mathbf{i} \cdot \mathbf{r} + k. \quad (1)$$

If $\mathbf{c} = 0$, (1) may be reduced to $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = \text{constant}$. We now proceed to consider this type of equation.

4. *Central Conics.* Let Φ be a constant uniplanar dyadic in the equation $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = \text{constant}$. If the constant is not zero it may be absorbed by the dyadic Φ so that the equation becomes $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$, or $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$. In these equations, we take Φ to be self-conjugate, for

T 4.1. If Φ is an anti-self-conjugate dyadic, the product $\mathbf{r} \cdot \Phi \cdot \mathbf{r}$ is identically zero for all values of \mathbf{r} .

By T 2.6., $\Phi = \frac{1}{2}(\Phi - \Phi_c)$. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

By T 2.5., $\Phi = a\mathbf{i}'\mathbf{i} + b\mathbf{j}'\mathbf{j}$ can be expressed as

$$\Phi = \frac{1}{2}[a\mathbf{i}'\mathbf{i} + b\mathbf{j}'\mathbf{j} - a\mathbf{i}\mathbf{i}' - b\mathbf{j}\mathbf{j}'].$$

Then $(x\mathbf{i} + y\mathbf{j}) \cdot [a\mathbf{i}'\mathbf{i} + b\mathbf{j}'\mathbf{j} - a\mathbf{i}\mathbf{i}' - b\mathbf{j}\mathbf{j}'] \cdot (x\mathbf{i} + y\mathbf{j})$ gives

$$ax^2\mathbf{i} \cdot \mathbf{i}' + by^2\mathbf{j} \cdot \mathbf{j}' - ax^2\mathbf{i}' \cdot \mathbf{i} - by^2\mathbf{j}' \cdot \mathbf{j} = 0.$$

T 4.2. If $\mathbf{r}' = \Phi \cdot \mathbf{r}$ and \mathbf{r} is allowed to generate a circle, \mathbf{r}' generates a central conic.

$$\mathbf{r}' = \Phi \cdot \mathbf{r}; \quad \mathbf{r} = \Phi^{-1} \cdot \mathbf{r}'; \quad \mathbf{r} = \mathbf{r}' \cdot \Phi_c^{-1}$$

$$\mathbf{r} \cdot \mathbf{r} = 1 = \mathbf{r}' \cdot (\Phi_c^{-1} \cdot \Phi^{-1}) \cdot \mathbf{r}' = \mathbf{r}' \cdot \Psi \cdot \mathbf{r}'. \quad \Psi = (\Phi_c^{-1} \cdot \Phi^{-1}) = a\mathbf{i}\mathbf{i} + b\mathbf{j}\mathbf{j}.$$

Let $\mathbf{r}' = x\mathbf{i} + y\mathbf{j}$; then $(x\mathbf{i} + y\mathbf{j}) \cdot (a\mathbf{i}\mathbf{i} + b\mathbf{j}\mathbf{j}) \cdot (x\mathbf{i} + y\mathbf{j}) = 1$ reduces to $ax^2 + by^2 = 1$, a central conic.

Let $\Phi = \pm \frac{\mathbf{i}\mathbf{i}}{a^2} \pm \frac{\mathbf{j}\mathbf{j}}{b^2}$, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$, gives two lines, real or imaginary; $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ gives $\pm \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$, an ellipse, real or imaginary, or an hyperbola.

T 4.3. $\Phi \cdot \mathbf{r}$ is normal to the central conic at \mathbf{r} .

The total differential of $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is $d\mathbf{r} \cdot \Phi \cdot \mathbf{r} + \mathbf{r} \cdot \Phi \cdot d\mathbf{r} = 0$; $2 d\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$; $d\mathbf{r} \cdot (\Phi \cdot \mathbf{r}) = 0$. Since $d\mathbf{r}$ is along the tangent, the theorem is proved.

T 4.3.1. The magnitude of the normal, i.e., $\Phi \cdot \mathbf{r}$, at \mathbf{r} , is equal to the reciprocal of the distance from the origin to the tangent at \mathbf{r} .

Let $\mathbf{n} = \Phi \cdot \mathbf{r}$. $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$; $\mathbf{r} \cdot \mathbf{n} = 1$; $\mathbf{r} \cdot \mathbf{n}\mathbf{n}_0 = 1$, where \mathbf{n}_0 is a unit vector along \mathbf{n} . $\mathbf{r} \cdot \mathbf{n}_0 = \mathbf{p}$; $\frac{1}{\mathbf{p}} = \mathbf{n}$.

T 4.4. The vector radius of a circle is perpendicular to the vector tangent.

Let \mathbf{p} be the vector from the center of the circle perpendicular to the vector tangent. Then $\mathbf{r} \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{p}$, since \mathbf{r} can not vary in magnitude, $\mathbf{r} = \mathbf{p}$.

T 4.4.1. For a circle

$$\Phi = \frac{\mathbf{i}\mathbf{i}}{a^2} + \frac{\mathbf{j}\mathbf{j}}{a^2}.$$

Since $\mathbf{r} = \mathbf{p}$ and \mathbf{p} is parallel to \mathbf{n} ,

$$\Phi \cdot \mathbf{r} = k\mathbf{r}; \quad \left(\frac{\mathbf{i}\mathbf{i}}{a^2} + \frac{\mathbf{j}\mathbf{j}}{b^2} \right) \cdot (x\mathbf{i} + y\mathbf{j}) = kx\mathbf{i} + ky\mathbf{j} \quad \text{then} \quad a = b.$$

T 4.5. The equation $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 1$ is that of the polar line of the point \mathbf{a} with respect to the central conic determined by Φ .

The proof of this theorem is like that for T 3.2.

T 4.6. The locus of the mid-points of a system of parallel chords of an ellipse is a straight line passing through the center of the ellipse.

Let \mathbf{s} be a vector to the mid-points of a system of parallel chords parallel to \mathbf{a} ,

then $\mathbf{r} = \mathbf{s} + x\mathbf{a}$ and $(\mathbf{s} + x\mathbf{a}) \cdot \Phi \cdot (\mathbf{s} + x\mathbf{a}) = 1$ gives $\mathbf{s} \cdot \Phi \cdot \mathbf{s} + 2x\mathbf{s} \cdot \Phi \cdot \mathbf{a} + x^2\mathbf{a} \cdot \Phi \cdot \mathbf{a} = 1$, a quadratic in x .

The stated conditions make $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 0$ for any one, and thus, with \mathbf{s} a variable, all of the chords parallel to \mathbf{a} . This is the equation of a straight line passing through the origin.

T 4.6.1. The locus of the mid-points of the chords parallel to \mathbf{s} falls upon that member of the \mathbf{a} system which passes through the origin.

The new locus is $\mathbf{s} \cdot \Phi \cdot \mathbf{s}' = 0$ with \mathbf{s}' variable. This is satisfied by $\mathbf{s}' = \mathbf{a}$.

We thus have a reciprocal relationship of the two systems of parallel lines.

D 4.2. The \mathbf{r}_1 and \mathbf{r}_2 members of these respective systems, are called *conjugate radii*, and thus we have,

T 4.6.2. $\mathbf{r}_1 \cdot \Phi \cdot \mathbf{r}_2 = 0$.

\mathbf{r}_1 is perpendicular to $\Phi \cdot \mathbf{r}_2$ and \mathbf{r}_2 is perpendicular to $\mathbf{r}_1 \cdot \Phi$. By T 4.3. $\mathbf{r}_1 \cdot \Phi$ and $\Phi \cdot \mathbf{r}_2$ are perpendicular to the tangents at \mathbf{r}_1 and \mathbf{r}_2 , respectively. Consequently \mathbf{r}_2 and \mathbf{r}_1 are parallel to the tangents at \mathbf{r}_1 and \mathbf{r}_2 respectively.

T 4.7. The equation of the line through two conjugate radii \mathbf{r}_1 , and \mathbf{r}_2 of an ellipse is $\mathbf{s} \cdot \Phi \cdot (\mathbf{r}_1 + \mathbf{r}_2) = 1$.

The equations of the tangents at \mathbf{r}_1 and \mathbf{r}_2 are $\mathbf{s} \cdot \Phi \cdot \mathbf{r}_1 = 1$ and $\mathbf{s}_2 \cdot \Phi \cdot \mathbf{r}_2 = 1$, respectively. Since the polar line of any point on the tangents passes through the point of tangency, the point of intersection of the two tangents is the pole of the line through \mathbf{r}_1 and \mathbf{r}_2 . The equation of the line is $\mathbf{s} \cdot \Phi \cdot \mathbf{a}' = 1$ where \mathbf{a}' is the vector to the pole and \mathbf{s} the vector to any point in the line. But $\mathbf{a}' = \mathbf{r}_1 + \mathbf{r}_2$, since \mathbf{r}_1 and \mathbf{r}_2 and the two tangents form a parallelogram. Then $\mathbf{s} \cdot \Phi \cdot (\mathbf{r}_1 + \mathbf{r}_2) = 1$ becomes the equation of the said line.

T 4.8. If $\mathbf{r}' = \Phi \cdot \mathbf{r}$, where $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is an ellipse or hyperbola, \mathbf{r}' , as a position vector, describes an ellipse or hyperbola, respectively.

$\mathbf{r}' = \Phi \cdot \mathbf{r}$, then $\mathbf{r} = \mathbf{r}' \cdot \Phi^{-1}$ and $\mathbf{r}' \cdot \Phi^{-1} \cdot \Phi \cdot \mathbf{r}' \cdot \Phi^{-1} = 1$; but $\mathbf{r}' \cdot \Phi^{-1} \cdot \Phi \cdot \Phi^{-1} \cdot \mathbf{r}' = 1$ gives $\mathbf{r}' \cdot \Phi^{-1} \cdot \mathbf{r}' = 1$.

T 4.9. The polar lines of all points on a line through the center of a central conic are parallel.

$$\Phi \cdot t\mathbf{r}_0 = t(\Phi \cdot \mathbf{r}_0).$$

$\mathbf{s} \cdot \Phi \cdot t\mathbf{r}_0 = 1$ gives lines perpendicular to $\Phi \cdot \mathbf{r}_0$ for all values of t , cutting $\Phi \cdot \mathbf{r}_0$ at a distance $\frac{1}{t}$ from the origin. Thus $\mathbf{s} \cdot \Phi \cdot t\mathbf{r}_0 = 1$ is a family of parallel lines, with parameter t .

We state two special cases of T 4.9. the proofs of which are obvious.

The polar line of a pole is parallel to the tangent which is drawn at the point where the vector from the center of the conic to the pole intersects the conic.

When $\mathbf{a} = \mathbf{r}$, the polar line becomes identical with the vector tangent at the terminus of \mathbf{a} .

T 4.10. Any secant line AB is the polar of the point \mathbf{a} determined by the tangents at A and B.

We shall designate A and B by \mathbf{r}' and \mathbf{r}'' , respectively. Then (1) $\mathbf{s}' \cdot \Phi \cdot \mathbf{r}' = 1$; (2) $\mathbf{s}'' \cdot \Phi \cdot \mathbf{r}'' = 1$; and (3) $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 1$. $\mathbf{s}' = \mathbf{s}'' = \mathbf{a}$ satisfies (1) and (2), but

$\mathbf{s} = \mathbf{r}'$ and $\mathbf{s} = \mathbf{r}''$ satisfies (3) which makes the secant through \mathbf{r}' and \mathbf{r}'' the polar of \mathbf{a} .

T 4.11. The limit of the polar of $\mathbf{a} = k\mathbf{i} + m\mathbf{j}$, when m becomes large and k remains some fixed constant, is $y = 0$, or $\mathbf{s} = x\mathbf{i}$.

$$\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 1;$$

$$(xi + yj) \cdot \Phi \cdot (ki + mj) = 1;$$

$$\lim_{m \rightarrow \infty} \left(\frac{kx}{a^2} + \frac{my}{b^2} = 1 \right) = \lim_{m \rightarrow \infty} \left(\frac{kx}{ma^2} + \frac{y}{b^2} = \frac{1}{m} \right);$$

$$\frac{y}{b^2} = 0; \quad y = 0; \quad \mathbf{s} = x\mathbf{i}.$$

T 4.12. If $\Phi = \frac{ii}{a^2} + \frac{jj}{b^2}$, $\Psi = \frac{bij}{a} - \frac{aji}{b}$, and $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$, $\mathbf{r} \cdot \Psi$ is perpendicular to $\Phi \cdot \mathbf{r}$.

$$\mathbf{r} \cdot \Psi = -\frac{ayj}{b} + \frac{xbj}{a}; \quad \Phi \cdot \mathbf{r} = \frac{x\mathbf{i}}{a^2} + \frac{y\mathbf{j}}{b^2};$$

$$(\mathbf{r} \cdot \Psi) \cdot (\Phi \cdot \mathbf{r}) = -\frac{xy}{ab} + \frac{xy}{ab} = 0.$$

T 4.12.1. $\mathbf{r} \cdot \Psi$ is the radius conjugate to \mathbf{r} in the ellipse $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$.

$\Phi \cdot \mathbf{r}$ is perpendicular to the radius vector conjugate to \mathbf{r} since it is perpendicular to the tangent at the terminus of \mathbf{a} which is parallel to the conjugate of \mathbf{a} . We need, then, only to check for length.

$$(\mathbf{r} \cdot \Psi) \cdot \Phi \cdot (\mathbf{r} \cdot \Psi) \stackrel{?}{=} 1,$$

$$\left(-\frac{ayi}{b} + \frac{xbj}{a} \right) \cdot \Phi \cdot \left(-\frac{ayi}{b} + \frac{xbj}{a} \right) \stackrel{?}{=} 1; \quad \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1.$$

T 4.13. The polar line of a point at infinity in the direction \mathbf{a} is the same as the diameter conjugate with \mathbf{a} .

In $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 1$ let us substitute $y\mathbf{a}$ in place of \mathbf{a} , where y is a scalar constant. Then $\mathbf{s} \cdot \Phi \cdot y\mathbf{a} = 1$; or $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = \frac{1}{y}$. When $y \rightarrow \infty$, $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 0$. By T 4.6.1., this is the equation of a radius \mathbf{s} conjugate with \mathbf{a} .

T 4.14. If $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is a circle, i.e., $\Phi = \frac{ii}{a^2} + \frac{jj}{b^2}$, the locus of the terminus of the normal vector $\Phi \cdot \mathbf{r}$ is a circle concentric with $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$.

By T 4.4. $\Phi \cdot \mathbf{r}$ and \mathbf{r} are collinear; or $\Phi \cdot \mathbf{r} + \mathbf{r} = m\mathbf{r}$. But $\sqrt{\Phi \cdot \mathbf{r} \cdot \Phi \cdot \mathbf{r}} = \frac{1}{p} = \frac{1}{a}$ and $r = a$. Consequently the locus of $\left(a + \frac{1}{a}\right)\mathbf{r}_0$, as $a\mathbf{r}_0$ generates a circle, is a circle concentric with $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$, and with equation $\mathbf{r}' \cdot \Psi \cdot \mathbf{r}' = 1$, where $\Psi = \frac{ii}{\left(a + \frac{1}{a}\right)^2} + \frac{jj}{\left(a + \frac{1}{a}\right)^2}$, and $\mathbf{r}' = \left[\left(a + \frac{1}{a}\right)\mathbf{r}_0\right]$.

T 4.15. The normal at any point on an ellipse, $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$, makes equal angles with the lines joining that point to the foci.

The position vectors of the foci are $\mathbf{f} = a\mathbf{e}_1$ and $\mathbf{f}' = -a\mathbf{e}_1$. The vectors joining the foci to the point in question are, $\mathbf{r} + a\mathbf{e}_1$ and $\mathbf{r} - a\mathbf{e}_1$. Unit vectors along these are

$$\frac{\mathbf{r} + a\mathbf{e}_1}{\sqrt{(\mathbf{r} + a\mathbf{e}_1) \cdot (\mathbf{r} + a\mathbf{e}_1)}}$$

and

$$\frac{\mathbf{r} - a\mathbf{e}_1}{\sqrt{(\mathbf{r} - a\mathbf{e}_1) \cdot (\mathbf{r} - a\mathbf{e}_1)}}$$

respectively. But $(a + ex) = \sqrt{(\mathbf{r} + a\mathbf{e}_1) \cdot (\mathbf{r} + a\mathbf{e}_1)}$, and $(a - ex) = \sqrt{(\mathbf{r} - a\mathbf{e}_1) \cdot (\mathbf{r} - a\mathbf{e}_1)}$ by the focus-directrix-eccentricity definition of the ellipse.

The equality of

$$\frac{\mathbf{r} + a\mathbf{e}_1}{a + ex} \cdot \Phi \cdot \mathbf{r} = \frac{\mathbf{r} - a\mathbf{e}_1}{a - ex} \cdot \Phi \cdot \mathbf{r}, \quad \frac{1 + \frac{ex}{a}}{a + ex} = \frac{1 - \frac{ex}{a}}{a - ex}, \quad \frac{1}{a} = \frac{1}{a}$$

establishes the truth of the theorem.

T 4.16. The normal at any point of the hyperbola $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ makes equal angles with the lines joining the point to the foci of the hyperbola.

We must merely test the equality of

$$\frac{(\mathbf{r} + a\mathbf{e}_1) \cdot \Phi \cdot \mathbf{r}}{\sqrt{(\mathbf{r} + a\mathbf{e}_1) \cdot (\mathbf{r} + a\mathbf{e}_1)}} \stackrel{?}{=} \frac{(a\mathbf{e}_1 - \mathbf{r}) \cdot \Phi \cdot \mathbf{r}}{\sqrt{(a\mathbf{e}_1 - \mathbf{r}) \cdot (a\mathbf{e}_1 - \mathbf{r})}}.$$

Substituting

$$ex + a = \sqrt{(\mathbf{r} + a\mathbf{e}_1) \cdot (\mathbf{r} + a\mathbf{e}_1)} \quad \text{and} \quad ex - a = \sqrt{(a\mathbf{e}_1 - \mathbf{r}) \cdot (a\mathbf{e}_1 - \mathbf{r})},$$

we have

$$\frac{(\mathbf{r} + a\mathbf{e}_1) \cdot \Phi \cdot \mathbf{r}}{ex + a} \stackrel{?}{=} \frac{(a\mathbf{e}_1 - \mathbf{r}) \cdot \Phi \cdot \mathbf{r}}{ex - a}; \quad \frac{1 + ex}{a(1 + ex)} \stackrel{?}{=} \frac{ex - a}{a(ex - a)}, \quad \frac{1}{a} = \frac{1}{a},$$

which proves the theorem.

T 4.17. The necessary and sufficient conditions that two conics $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ and $\mathbf{r} \cdot \Psi \cdot \mathbf{r} = 1$ be confocal is that Φ^{-1} and Ψ^{-1} shall differ only by a scalar multiple of the idemfactor.

Let $\Phi = \frac{ii}{a^2} + \frac{jj}{b^2}$ and $\Psi = \frac{ii}{n^2} + \frac{jj}{m^2}$. Then

$$\Phi^{-1} = a^2 ii + b^2 jj; \quad \Psi^{-1} = n^2 ii + m^2 jj.$$

$$\Phi^{-1} + \lambda I = \Psi^{-1}, \quad \Phi^{-1} = -\lambda I + \Psi^{-1}$$

where λ is a scalar constant.

$$\Phi = \frac{1}{n^2 - \lambda} \mathbf{ii} + \frac{1}{m^2 - \lambda} \mathbf{jj},$$

then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ gives $\frac{x^2}{n^2 - \lambda} + \frac{y^2}{m^2 - \lambda} = 1$.

Now we shall test any two of the set of conics produced as λ varies, say $\lambda = \lambda_1$; and $\lambda = \lambda_2$.

$$\Phi = \frac{\mathbf{ii}}{n^2 - \lambda_1} + \frac{\mathbf{jj}}{m^2 - \lambda_1}; \quad \Psi = \frac{\mathbf{ii}}{n^2 - \lambda_2} + \frac{\mathbf{jj}}{m^2 - \lambda_2};$$

$$\Phi^{-1} = (n^2 - \lambda_1) \mathbf{ii} + (m^2 - \lambda_1) \mathbf{jj}; \quad \Psi^{-1} = (n^2 - \lambda_2) \mathbf{ii} + (m^2 - \lambda_2) \mathbf{jj};$$

$$\Phi^{-1} - \Psi^{-1} = (\lambda_2 - \lambda_1) \mathbf{ii} - (\lambda_2 - \lambda_1) \mathbf{jj};$$

$$\Phi^{-1} - \Psi^{-1} = (\lambda_2 - \lambda_1) (\mathbf{ii} + \mathbf{jj}).$$

T 4.18. The sum of the squares of a pair of conjugate radii \mathbf{a}' and \mathbf{b}' of an ellipse, $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$, is constant.

We wish to prove $\mathbf{a}' \cdot \mathbf{a}' + \mathbf{b}' \cdot \mathbf{b}' = \text{constant}$.

$$\mathbf{a}' \cdot \mathbf{a}' + \mathbf{b}' \cdot \mathbf{b}' = \mathbf{r} \cdot \mathbf{r} + (\mathbf{r} \cdot \Psi) \cdot (\mathbf{r} \cdot \Psi);$$

$$\mathbf{a}' = \mathbf{r} \text{ and } \mathbf{b}' = \mathbf{r} \cdot \Psi;$$

$$\mathbf{r} \cdot \mathbf{r} = x^2 + y^2 \text{ and } \mathbf{r} \cdot \Phi \cdot \mathbf{r} = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

$$y^2 = \frac{b^2}{a^2} (a^2 - x^2);$$

$$\mathbf{r} \cdot \mathbf{r} = b^2 + \frac{a^2 - b^2}{a^2} x^2 = b^2 + e^2 x^2;$$

$$(\mathbf{r} \cdot \Psi) \cdot (\mathbf{r} \cdot \Psi) = \frac{a^2 y^2}{b^2} + \frac{b^2}{a^2} x^2;$$

but

$$\frac{a^2}{b^2} y^2 = a^2 - x^2;$$

hence,

$$(\mathbf{r} \cdot \Psi) \cdot (\mathbf{r} \cdot \Psi) = a^2 - x^2 + \frac{b^2 x^2}{a^2} = a^2 - \left(1 - \frac{b^2}{a^2}\right) x^2;$$

$$= a^2 - \left(\frac{a^2 - b^2}{a^2}\right) x^2;$$

$$(\mathbf{r} \cdot \Psi) \cdot (\mathbf{r} \cdot \Psi) = a^2 - e^2 x^2.$$

Thus,

$$\mathbf{r} \cdot \mathbf{r} + (\mathbf{r} \cdot \Psi) \cdot (\mathbf{r} \cdot \Psi) = a^2 + b^2.$$

An alternate proof for T 4.18.

We wish to prove $\mathbf{a}' \cdot \mathbf{a}' + \mathbf{b}' \cdot \mathbf{b}'$ constant.

By theorem 2.20, we may express \mathbf{r} in terms of the sine and cosine of an angle s ;

$$\mathbf{r} = a\mathbf{i} \cos s + b\mathbf{j} \sin s;$$

$$\frac{a\mathbf{r}}{as} = a\mathbf{i} \sin s + b\mathbf{j} \cos s;$$

$$\mathbf{b}' = m \frac{d\mathbf{r}}{ds}; \quad \mathbf{b}' = m(a\mathbf{i} \sin s + b\mathbf{j} \cos s);$$

$$m(-a\mathbf{i} \sin s + b\mathbf{j} \cos s) \cdot \Phi \cdot m(-a\mathbf{i} \sin s + b\mathbf{j} \cos s) = 1;$$

$$m^2(\sin^2 s + \cos^2 s) = 1; m^2 = 1; m = \pm 1;$$

$$\mathbf{a}' \cdot \mathbf{a}' = a^2 \cos^2 s + b^2 \sin^2 s; \quad (1)$$

$$\mathbf{b}' \cdot \mathbf{b}' = a^2 \sin^2 s + b^2 \cos^2 s; \quad (2)$$

Adding (1) and (2) we have $\mathbf{a}' \cdot \mathbf{a}' + \mathbf{b}' \cdot \mathbf{b}' = a^2 + b^2$.

T 4.19. The length of the latus rectum of an ellipse is $\frac{2b^2}{a}$. $\mathbf{r} = ae\mathbf{i} + n\mathbf{j}$ makes the latus rectum equal $2n$.

$$(ae\mathbf{i} + n\mathbf{j}) \cdot \Phi \cdot (ae\mathbf{i} + n\mathbf{j}) = 1;$$

$$e^2 + \frac{n^2}{b^2} = 1; \quad n^2 = b^2(1 - e^2);$$

$$n = b\sqrt{1 - e^2};$$

$$2n = 2b\sqrt{1 - e^2};$$

but

$$b^2 = a^2(1 - e^2); \quad \frac{b}{a} = \sqrt{1 - e^2};$$

$$2n = \frac{2b^2}{a}.$$

T 4.20. If two confocal central conics intersect they do so at right angles.

Let the conics be $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ and $\mathbf{r} \cdot \Psi \cdot \mathbf{r} = 1$. Let $\mathbf{n} = \Phi \cdot \mathbf{r}$, and $\mathbf{n}_1 = \Psi \cdot \mathbf{r}$. Then, $\mathbf{r} = \Phi^{-1} \cdot \mathbf{n}$ and $\mathbf{r} = \Psi^{-1} \cdot \mathbf{n}_1$. Since the conics are confocal, $\Phi^{-1} - \Psi^{-1} = \lambda \mathbf{I}$.

Then

$$\mathbf{r} \cdot \Phi \cdot \mathbf{r} = \Phi^{-1} \cdot \mathbf{n} \cdot \Phi \cdot \Phi^{-1} \cdot \mathbf{n} = 1; \mathbf{n} \cdot \Phi^{-1} \cdot \mathbf{n} = 1;$$

$$\mathbf{r} \cdot \Psi \cdot \mathbf{r} = \Psi^{-1} \cdot \mathbf{n}_1 \cdot \Psi \cdot \Psi^{-1} \cdot \mathbf{n}_1 = 1;$$

$$\mathbf{n}_1 \cdot \Psi^{-1} \cdot \mathbf{n}_1 = 1;$$

$$\mathbf{r} = \Psi^{-1} \cdot \mathbf{n}_1 = \Phi^{-1} \cdot \mathbf{n} = (\Psi^{-1} + \lambda \mathbf{I}) \cdot \mathbf{n} = \Psi^{-1} \cdot \mathbf{n} + \lambda \mathbf{n};$$

$$\lambda \mathbf{n} = \Psi^{-1} \cdot \mathbf{n}_1 - \Psi^{-1} \cdot \mathbf{n};$$

$$\lambda \mathbf{n} \cdot \mathbf{n}_1 = 1 - \mathbf{n}_1 \cdot \Psi^{-1} \cdot \mathbf{n}. \quad (1)$$

Or

$$\begin{aligned} \mathbf{r} &= \Phi^{-1} \cdot \mathbf{n} = \Psi^{-1} \cdot \mathbf{n}_1 = (\Phi^{-1} - \lambda \mathbf{I}) \cdot \mathbf{n}_1 = \Phi^{-1} \cdot \mathbf{n}_1 - \lambda \mathbf{n}_1; \\ -\lambda \mathbf{n}_1 &= \Phi^{-1} \cdot \mathbf{n} - \Phi^{-1} \cdot \mathbf{n}_1; \\ \lambda \mathbf{n}_1 \cdot \mathbf{n} &= \mathbf{n} \cdot \Phi^{-1} \cdot \mathbf{n}_1 - \mathbf{n} \cdot \Phi^{-1} \cdot \mathbf{n}_1; \\ \lambda \mathbf{n}_1 \cdot \mathbf{n} &= \mathbf{n} \cdot \Phi^{-1} \cdot \mathbf{n}_1 - 1. \end{aligned} \quad (2)$$

(1) + (2) gives $\lambda \mathbf{n}_1 \cdot \mathbf{n} = 0$, which proves the theorem.

T 4.21. If $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is the equation of an hyperbola, the equation of the tangents at infinity, i.e., the asymptotes, are represented by $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$.

Let $\mathbf{r} = m\mathbf{r}_0$ be a point on the hyperbola, where \mathbf{r}_0 is a unit vector.

The equation $\mathbf{s} \cdot \Phi \cdot m\mathbf{r}_0 = 1$ is that of the tangent line of $m\mathbf{r}_0$. $\lim_{m \rightarrow \infty} (\mathbf{s} \cdot \Phi \cdot m\mathbf{r}_0 = 1)$ is the $\lim_{m \rightarrow \infty} \left(\mathbf{s} \cdot \Phi \cdot \mathbf{r}_0 = \frac{1}{m} \right)$, or $\mathbf{s} \cdot \Phi \cdot \mathbf{r}_0 = 0$.

Since \mathbf{r}_0 has the direction of \mathbf{r} , and \mathbf{s} is perpendicular to $\Phi \cdot \mathbf{r}$, $\mathbf{s} \cdot \Phi \cdot \mathbf{r} = 0$, which is a line through the origin. In this line $\mathbf{s} = t\mathbf{r}$, hence $t\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$, or $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$, which is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

T 4.22. The two segments of a secant line parallel to \mathbf{a} , intercepted between the asymptotes and the hyperbola are equal.

Let \mathbf{b} be a vector from the origin to the mid-point of that part of \mathbf{a} which is the chord falling on the given secant.

$\mathbf{r} = \mathbf{b} + x\mathbf{a}$ is the equation of the secant.

$\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is that of the hyperbola.

$(\mathbf{b} + x\mathbf{a}) \cdot \Phi \cdot (\mathbf{b} + x\mathbf{a}) = 1$ gives $\mathbf{b} \cdot \Phi \cdot \mathbf{b} + 2x\mathbf{b} \cdot \Phi \cdot \mathbf{a} + x^2\mathbf{a} \cdot \Phi \cdot \mathbf{a} = 1$, whose solutions give the intersections of the secant with the hyperbola.

Since the two values of x are equal and opposite in sign, $\mathbf{b} \cdot \Phi \cdot \mathbf{a} = 0$. The common solution of the secant, $\mathbf{r} = \mathbf{b} + x'\mathbf{a}$, and the asymptotes, $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$, are involved in $(\mathbf{b} + x'\mathbf{a}) \cdot \Phi \cdot (\mathbf{b} + x'\mathbf{a}) = 0$, i.e., $\mathbf{b} \cdot \Phi \cdot \mathbf{b} + 2x'\mathbf{b} \cdot \Phi \cdot \mathbf{a} + x'^2\mathbf{a} \cdot \Phi \cdot \mathbf{a} = 0$.

But $\mathbf{b} \cdot \Phi \cdot \mathbf{a} = 0$, therefore the coefficient of $2x'$ vanishes, which means the two roots of this equation in x' are equal in magnitude and opposite in sign.

T 4.23. The polar \mathbf{a}' of the secant through the radii \mathbf{a} and \mathbf{b} is $k(\mathbf{a} + \mathbf{b})$.

Let $\mathbf{r} = m\mathbf{a} + n\mathbf{b}$ be the pole of the line through \mathbf{a} and \mathbf{b} . Then, $\mathbf{s} \cdot \Phi \cdot (m\mathbf{a} + n\mathbf{b}) = 1$, and $\mathbf{a} \cdot \Phi \cdot (m\mathbf{a} + n\mathbf{b}) = 1$ gives $m + n\mathbf{a} \cdot \Phi \cdot \mathbf{b} = 1$. Similarly $\mathbf{b} \cdot \Phi \cdot (m\mathbf{a} + n\mathbf{b}) = 1$ gives $n + m\mathbf{b} \cdot \Phi \cdot \mathbf{a} = 1$. Subtracting, $(m - n) + \mathbf{b} \cdot \Phi \cdot \mathbf{a} (n - m) = 0$, $(m - n)(1 - \mathbf{b} \cdot \Phi \cdot \mathbf{a}) = 0$. Either $m - n = 0$, or $\mathbf{b} \cdot \Phi \cdot \mathbf{a} = 1$. The latter is false unless $\mathbf{a} = \mathbf{b}$. Thus $m = n$, and $\mathbf{a}' = k(\mathbf{a} + \mathbf{b})$.

T 4.23.1. $(\mathbf{a} + \mathbf{b}) \cdot \Phi \cdot (\mathbf{s} - \mathbf{a}) = 0$ is the equation of the secant determined by radii, \mathbf{a} and \mathbf{b} .

$$\begin{aligned} \mathbf{s} \cdot \Phi \cdot k(\mathbf{a} + \mathbf{b}) &= 1, \\ k d\mathbf{s} \cdot \Phi \cdot (\mathbf{a} + \mathbf{b}) &= 0, \\ d\mathbf{s} \cdot \Phi \cdot (\mathbf{a} + \mathbf{b}) &= 0. \end{aligned}$$

Thus ds is perpendicular to $\Phi \cdot (a + b)$. Since ds is along the secant we may substitute $s - a$ for ds . Thus, $(a + b) \cdot \Phi \cdot (s - a) = 0$.

T 4.23.2. The equation of the tangent at a is $s \cdot \Phi \cdot a = 1$.

Let b approach a as a limit; then

$$(a + a) \cdot \Phi \cdot (s - a) = 0;$$

$$2 s \cdot \Phi \cdot a - 2 a \cdot \Phi \cdot a = 0; s \cdot \Phi \cdot a = 1.$$

5. *Quadric Surfaces.* The proofs of the theorems on conics can be modified to establish the analogous theorems in space of three dimensions. We need to make uniplanar Φ for the parabola, planar, planar Φ of central conics, complete, and to set $r = xi + yj + zk$. The following theorems will be numbered to correspond with the similar theorems of sections three and four. Thus, T 5.3.1. will correspond to T 3.1., and T 5.4.1. will correspond to T 4.1.

Proofs will be given only when they differ sufficiently from the proofs in two dimensions to be of especial interest.

T 3.5.1. If Φ is uniplanar, r a variable vector, v any constant vector, and k any scalar, then $r \cdot \Phi \cdot r = v \cdot I \cdot r + k$ becomes

$$a_{11}x^2 \pm a_{22}y^2 = ax + by + cz + k,$$

a non-central quadric.

In the next theorem, let

$$\Phi = \pm \frac{ii}{a^2} \pm \frac{jj}{b^2}, \quad v = 2ck \quad \text{and} \quad k = 0.$$

T 5.3.2. The equation of the polar plane of the point determined by the vector a is $s \cdot \Phi \cdot a = 2 ck \cdot I \cdot \frac{(s + a)}{2}$.

If the signs of the two dyads of Φ are different and c is positive, $r \cdot \Phi \cdot r = 2ck \cdot I \cdot r$ gives an hyperbolic paraboloid; if the signs of Φ are alike, an elliptic paraboloid.

If $\Phi = \frac{ii}{4a} + \frac{jj}{4a}$, then $r \cdot \Phi \cdot r = k \cdot I \cdot r$ becomes $x^2 + y^2 = 4az$, a paraboloid of revolution.

T 5.3.2.1. The reciprocal relationship; if b is on the polar of a , a is on the polar of b , may be readily established.

T 5.3.2.2. The polar of a point a on the paraboloid is the tangent plane to the paraboloid at that point.

T 5.3.3. The locus of the mid-points of a system of parallel chords of a paraboloid is a plane parallel to the axis.

T 5.4.1. If Φ is an anti-self-conjugate dyadic, the product of $r \cdot \Phi \cdot r$ is identically zero for all values of r .

T 5.4.2. If $r' = \Phi \cdot r$ and r is allowed to generate a sphere, r' generates a central quadric.

$$\text{If } \Phi = \pm \frac{ii}{a^2} \pm \frac{jj}{b^2} \pm \frac{kk}{c^2} \text{ where any combinations of signs may occur, } r \cdot \Phi \cdot r = 1$$

and $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$ give rise to the central quadric surfaces; the real ellipsoid, the hyperboloid of one sheet, an hyperboloid of two sheets, the imaginary ellipsoid, and the real or imaginary cone.

T 5.4.3. $\Phi \cdot \mathbf{r}$ is normal to the tangent plane to the central quadric.

T 5.4.3.1. The magnitude of the normal \mathbf{n} , at \mathbf{r} , i.e., $\Phi \cdot \mathbf{r}$, is equal to the reciprocal of the distance from the origin to the tangent plane at \mathbf{r} .

T 5.4.4. The radius vector of a sphere is perpendicular to the tangent plane.

T 5.4.5. In any central quadric $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 1$ is the equation of the polar plane of the pole \mathbf{a} with respect to that quadric.

T 5.4.6. The locus of the mid-points of a system of parallel chords of any quadric is a plane. For central quadrics this plane passes through the center.

This plane is called the diametral plane conjugate with the system of chords. It is parallel to the plane drawn tangent to the ellipsoid at the extremity of that one of the chords which passes through the center. This theorem leads to a system of three radii, conjugate in pairs.

T 5.4.7. The equation of the plane determined by three conjugate radii, \mathbf{a} , \mathbf{b} , and \mathbf{c} is $\mathbf{r} \cdot \Phi \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) = 1$.

From the pole and polar relationships involved, \mathbf{a} , \mathbf{b} , and \mathbf{c} determine six planes, three diametral and three tangent, which determine a parallelepiped with a vertex $\mathbf{a} + \mathbf{b} + \mathbf{c}$. This vertex has for its polar, $\mathbf{r} \cdot \Phi \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$, which passes through \mathbf{a} , \mathbf{b} , and \mathbf{c} .

T 5.4.8. If $\mathbf{r}' = \Phi \cdot \mathbf{r}$, where $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is an ellipsoid or hyperboloid, \mathbf{r}' , as a position vector describes an ellipsoid or hyperboloid, respectively.

T 5.4.9. The polar planes of all points on a line through the center of a central quadric are parallel.

T 5.4.10. Any secant plane is the polar of the point \mathbf{a} , determined by the tangent cone of the conic of intersection.

T 5.4.11. The polar of any point in a plane passes through the pole of that plane.

T 5.4.15. The normal at any point on a central quadric, $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$, makes equal angles with the lines joining that point to the foci of the quadric.

T 5.4.20. The necessary and sufficient condition that two quadrics whose equations are, $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ and $\mathbf{r} \cdot \Psi \cdot \mathbf{r} = 1$, shall be confocal is that Φ^{-1} and Ψ^{-1} shall differ by a scalar multiple of the idemfactor.¹⁷

The proof is as follows:

Let

$$\Psi = \frac{\mathbf{ii}}{n^2} + \frac{\mathbf{jj}}{m^2} + \frac{\mathbf{kk}}{p^2};$$

then $\Psi^{-1} = n^2 \mathbf{ii} + m^2 \mathbf{jj} + p^2 \mathbf{kk}$; $\Phi = -\lambda \mathbf{I} + \Psi^{-1}$; where λ is a scalar constant.

$$\Phi = \left(\frac{1}{n^2 - \lambda} \right) \mathbf{ii} + \left(\frac{1}{m^2 - \lambda} \right) \mathbf{jj} + \left(\frac{1}{p^2 - \lambda} \right) \mathbf{kk}.$$

$$\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1 \text{ gives } \frac{x^2}{n^2 - \lambda} + \frac{y^2}{m^2 - \lambda} + \frac{z^2}{p^2 - \lambda} = 1.$$

¹⁷ Wills, *op. cit.*, p. 165.

By T 2.9, as λ varies $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ gives a system of confocal quadrics. Let

$$\Phi = \frac{\mathbf{ii}}{n^2 - \lambda_1} + \frac{\mathbf{jj}}{m^2 - \lambda_1} + \frac{\mathbf{kk}}{p^2 - \lambda_1},$$

and

$$\Psi = \frac{\mathbf{ii}}{n^2 - \lambda_2} + \frac{\mathbf{jj}}{m^2 - \lambda_2} + \frac{\mathbf{kk}}{p^2 - \lambda_2};$$

then

$$\Phi^{-1} = (n^2 - \lambda_1)\mathbf{ii} + (m^2 - \lambda_1)\mathbf{jj} + (p^2 - \lambda_1)\mathbf{kk};$$

$$\Psi^{-1} = (n^2 - \lambda_2)\mathbf{ii} + (m^2 - \lambda_2)\mathbf{jj} + (p^2 - \lambda_2)\mathbf{kk};$$

$\Phi^{-1} - \Psi^{-1} = (\lambda_2 - \lambda_1)\mathbf{I}$, which establishes the theorem.

T 5.4.19. The sum of the squares of three conjugate radii \mathbf{a}' , \mathbf{b}' , and \mathbf{c}' of an ellipsoid, $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$, is constant.

T 4.5.21. If $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is the equation of an hyperboloid, the equation of the tangent cone at infinity, is $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$.

T 5.4.22. The two segments of a secant line parallel to \mathbf{a} intercepted between the cone $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$ and the hyperboloid $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ are equal.

6. *Concluding Remarks.* We refrained in the title of this paper from saying a study of conics; we are investigating a method and not conics. And now, as we come to the end of the discussion, we refrain from using the word conclusion. New and enticing topics present themselves and *commencement* might be a more appropriate word. Some topics for further study with the aid of the dyadic operator would be: the linear function in plane and solid analytic geometry, functions in four and higher dimensions, poles and polars, the principle of duality, and other topics of projective geometry. Here there is opportunity for further investigation and generalization in the spirit and method of the attack used in this paper.

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THE DYADIC OPERATOR AS APPLIED TO LINES, PLANES, AND QUADRIC SURFACES

LAURA LEE COWEN

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1. *Introduction.* This study is divided into two sections, 3 and 4 of this paper. In 3 we make use of the dyadic operator in both two and three dimensional space where lines and planes are studied; in 4 we apply the operator to a study of quadric surfaces. A study was carried out for conics by James Loyd Shawn at the North Texas State Teacher's College, under direction of Forbes B. Wiley. The Theorems that he suggested for the projection of his work into the third dimension are presented here in 4.

2. *Assumed Relationships.* It is assumed that the reader of this paper has a knowledge of *Vector Analysis* and thus that he has familiarity with the following: line-vectors, hereafter called vectors: \mathbf{a} , \mathbf{b} etc.; their magnitude: a , b etc., respectively; their algebraic sum: $\mathbf{a} + \mathbf{b} = \mathbf{c}$; the scalar product of two vectors $\mathbf{a} \cdot \mathbf{b}$; the vector product of two vectors: $\mathbf{a} \times \mathbf{b}$; the mutually perpendicular unit vectors: \mathbf{i} , \mathbf{j} , \mathbf{k} ; the position vector \mathbf{r} ; the derivative of a variable vector \mathbf{r} ; and the dyadic operator Φ .

3. Applications to Straight Lines and Planes.

3.1 If $\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk}$ is the idemfactor, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + \sqrt{c}\mathbf{k}$ is a variable vector to any point in a plane p , \mathbf{va} (augmented \mathbf{v}) $= a\mathbf{i} + b\mathbf{j} + \sqrt{c}\mathbf{k}$ is any constant vector to a fixed point in p , and $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is any constant vector parallel to the plane p , then $\mathbf{va} \cdot \mathbf{I} \cdot \mathbf{r} = 0$ is the equation of a straight line in the plane p .

This follows from the fact that $\mathbf{va} \cdot \mathbf{I} \cdot \mathbf{r} = 0$ becomes $ax + by + c = 0$, the cartesian equation of a straight line in plane p .

3.2 If $\mathbf{v}_l \times \mathbf{v}_m = 0$ and $\mathbf{va}_l \times \mathbf{va}_m \neq 0$; $l, m = 1$ to ∞ , then $\mathbf{va}_l \cdot \mathbf{I} \cdot \mathbf{r} = 0$; $l = 1$ to ∞ is the equation of a system of distinct parallel lines, with parameter l .

This follows from the fact that, if the cross product of the constant vectors of any two of these lines is zero, the coefficients of the x 's and y 's in the cartesian equations of those lines are proportional; and if the cross product of the augmented vector of any two of these lines is not zero, the lines are distinct.

3.3 If $\mathbf{v}_l \times \mathbf{v}_m \neq 0$, $l, m = 1$ to ∞ , then $\mathbf{va}_l \cdot \mathbf{I} \cdot \mathbf{r} = 0$; $l = 1$ to ∞ , are the equations of lines, no two of which are parallel.

This follows from the fact that if the cross product of the constant vectors of any two of these lines is not zero, the coefficients of the x 's and y 's in the cartesian equations of those lines are not proportional.

3.4 If $\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} + \mathbf{ll}$ is the four-dimensional¹ idemfactor, $\mathbf{r} = x\mathbf{i} +$

¹ A four-dimensional idemfactor means a set of four mutually perpendicular unit vectors.

$y\mathbf{j} + z\mathbf{k} + \sqrt{d}\mathbf{l}$ is a variable vector, \mathbf{va} (augmented \mathbf{v}) = $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} + \sqrt{d}\mathbf{l}$ is any constant vector, and $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is another constant vector, then $\mathbf{va} \cdot \mathbf{l} \cdot \mathbf{r} = 0$ is the equation of a plane.

This follows from the fact that $\mathbf{va} \cdot \mathbf{l} \cdot \mathbf{r} = 0$ becomes $ax + by + cz + d = 0$, the cartesian equation of a plane.

3.5 If $\mathbf{v}_l \times \mathbf{v}_m = 0$ and $\mathbf{va}_l \times \mathbf{va}_m \neq 0$; $l, m = 1$ to ∞ , then $\mathbf{va}_l \cdot \mathbf{l} \cdot \mathbf{r} = 0$ is the equation of a system of distinct parallel planes.

This follows from the fact that if the cross product of the constant vectors of any two of these planes is zero, the coefficients of the x 's, y 's, and z 's in the cartesian equations of those planes are proportional; and if the cross product of the augmented vector of any two of these planes is not zero, the planes are distinct.

3.6 If $\mathbf{v}_l \times \mathbf{v}_m \neq 0$; $l, m = 1$ to ∞ , then $\mathbf{va}_l \cdot \mathbf{l} \cdot \mathbf{r} = 0$ are the equations for planes, no two of which are parallel.

This follows from the fact that if the cross product of the constant vectors of any two of these planes is not zero, the coefficients of the x 's, y 's and z 's in the cartesian equations of these planes are not proportional.

3.7 If $\mathbf{r} = (x - \alpha)\mathbf{i} + (y - \beta)\mathbf{j} + (z - \gamma)\mathbf{k}$ is a variable vector, $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a constant vector, and \mathbf{l} is as in 3.1, then $\mathbf{v}_l \cdot \mathbf{l} \cdot \mathbf{r} = 0$; $l = 1$ to ∞ , is the equation of a bundle of planes, with parameter l .

This follows from the fact that $\mathbf{v}_l \cdot \mathbf{l} \cdot \mathbf{r} = 0$; $l = 1$ to ∞ becomes $a(x - \alpha) + b(y - \beta) + c(z - \gamma) = 0$, a bundle of planes through (α, β, γ) .

3.8 If $\mathbf{va}_l \times \mathbf{va}_m = 0$; $l, m = 1$ to ∞ then $\mathbf{va}_l \cdot \mathbf{l} \cdot \mathbf{r} = 0$, $i = 1$ to ∞ is the equation of a system of coincident planes, with parameter l .

This follows from the fact that if the cross product of the augmented vectors of any two of these planes is zero, the equation of one of these planes is a constant multiplier of the other plane.

3.9 If $\mathbf{v}_1 \times \mathbf{v}_2 \neq 0$ and k_1 and k_2 are arbitrary constants not both zero, then $(k_1\mathbf{va}_1 + k_2\mathbf{va}_2) \cdot \mathbf{l} \cdot \mathbf{r} = 0$ is the equation of a pencil of planes whose common line of intersection is $\begin{cases} \mathbf{va}_1 \cdot \mathbf{l} \cdot \mathbf{r} = 0 \\ \mathbf{va}_2 \cdot \mathbf{l} \cdot \mathbf{r} = 0 \end{cases}$

This follows from the fact that $(k_1\mathbf{va}_1 + k_2\mathbf{va}_2) \cdot \mathbf{l} \cdot \mathbf{r} = 0$ becomes $k_1(a_1x + b_1y + c_1z + d_1) + k_2(a_2x + b_2y + c_2z + d_2) = 0$ the cartesian equation of a pencil of planes.

3.10 If $\mathbf{v}_1, \mathbf{v}_2$ and \mathbf{v}_3 are such that $\mathbf{v}_1 \times \mathbf{v}_2 = 0$ and $\mathbf{v}_2 \times \mathbf{v}_3 \neq 0$ then 1) $\mathbf{va}_1 \cdot \mathbf{l} \cdot \mathbf{r} = 0$ and 2) $\mathbf{va}_2 \cdot \mathbf{l} \cdot \mathbf{r} = 0$ are the equations for two parallel planes and 3) $\mathbf{va}_3 \cdot \mathbf{l} \cdot \mathbf{r} = 0$ is a third cutting the first two.

This follows from the fact that if the cross product of the constant vectors in the first two planes is zero, the coefficients of the x 's, y 's, and z 's in the cartesian equations of those planes are proportional; and if the cross product of the constant vectors in the second and third plane is not zero, the coefficients of the x 's, y 's and z 's in the cartesian equations of those planes are not proportional.

4. Application to Quadric Surfaces.

4.1 Equation of non-central quadric.

If Φ is uniplanar, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is a variable vector, $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is any constant vector, and k is any scalar, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = \mathbf{v} \cdot \mathbf{l} \cdot \mathbf{r} + k$ is the equation of a non-central quadric.

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This follows from the fact that $\mathbf{va} \cdot \mathbf{I} \cdot \mathbf{r} = 0$ becomes $ax + by + c = 0$, the cartesian equation of a straight line in plane p .

3.2 If $\mathbf{v}_l \times \mathbf{v}_m = 0$ and $\mathbf{va}_l \times \mathbf{va}_m \neq 0$; $l, m = 1$ to ∞ , then $\mathbf{va}_l \cdot \mathbf{I} \cdot \mathbf{r} = 0$; $l = 1$ to ∞ is the equation of a system of distinct parallel lines, with parameter l .

This follows from the fact that, if the cross product of the constant vectors of any two of these lines is zero, the coefficients of the x 's and y 's in the cartesian equations of those lines are proportional; and if the cross product of the augmented vector of any two of these lines is not zero, the lines are distinct.

3.3 If $\mathbf{v}_l \times \mathbf{v}_m \neq 0$, $l, m = 1$ to ∞ , then $\mathbf{va}_l \cdot \mathbf{I} \cdot \mathbf{r} = 0$; $l = 1$ to ∞ , are the equations of lines, no two of which are parallel.

This follows from the fact that if the cross product of the constant vectors of any two of these lines is not zero, the coefficients of the x 's and y 's in the cartesian equations of those lines are not proportional.

3.4 If $\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} + \mathbf{ll}$ is the four-dimensional¹ idemfactor, $\mathbf{r} = x\mathbf{i} +$

¹ A four-dimensional idemfactor means a set of four mutually perpendicular unit vectors.

$yj + zk + \sqrt{d}l$ is a variable vector, \mathbf{va} (augmented \mathbf{v}) = $ai + bj + ck + \sqrt{d}l$ is any constant vector, and $\mathbf{v} = ai + bj + ck$ is another constant vector, then $\mathbf{va} \cdot \mathbf{l} \cdot \mathbf{r} = 0$ is the equation of a plane.

This follows from the fact that $\mathbf{va} \cdot \mathbf{l} \cdot \mathbf{r} = 0$ becomes $ax + by + cz + d = 0$, the cartesian equation of a plane.

3.5 If $\mathbf{v}_l \times \mathbf{v}_m = 0$ and $\mathbf{va}_l \times \mathbf{va}_m \neq 0$; $l, m = 1$ to ∞ , then $\mathbf{va}_l \cdot \mathbf{l} \cdot \mathbf{r} = 0$ is the equation of a system of distinct parallel planes.

This follows from the fact that if the cross product of the constant vectors of any two of these planes is zero, the coefficients of the x 's, y 's, and z 's in the cartesian equations of those planes are proportional; and if the cross product of the augmented vector of any two of these planes is not zero, the planes are distinct.

3.6 If $\mathbf{v}_l \times \mathbf{v}_m \neq 0$; $l, m = 1$ to ∞ , then $\mathbf{va}_l \cdot \mathbf{l} \cdot \mathbf{r} = 0$ are the equations for planes, no two of which are parallel.

This follows from the fact that if the cross product of the constant vectors of any two of these planes is not zero, the coefficients of the x 's, y 's and z 's in the cartesian equations of these planes are not proportional.

3.7 If $\mathbf{r} = (x - \alpha)\mathbf{i} + (y - \beta)\mathbf{j} + (z - \gamma)\mathbf{k}$ is a variable vector, $\mathbf{v} = ai + bj + ck$ is a constant vector, and \mathbf{l} is as in 3.1, then $\mathbf{v}_l \cdot \mathbf{l} \cdot \mathbf{r} = 0$; $l = 1$ to ∞ , is the equation of a bundle of planes, with parameter l .

This follows from the fact that $\mathbf{v}_l \cdot \mathbf{l} \cdot \mathbf{r} = 0$; $l = 1$ to ∞ becomes $a(x - \alpha) + b(y - \beta) + c(z - \gamma) = 0$, a bundle of planes through (α, β, γ) .

3.8 If $\mathbf{va}_l \times \mathbf{va}_m = 0$; $l, m = 1$ to ∞ then $\mathbf{va}_l \cdot \mathbf{l} \cdot \mathbf{r} = 0$, $l = 1$ to ∞ is the equation of a system of coincident planes, with parameter l .

This follows from the fact that if the cross product of the augmented vectors of any two of these planes is zero, the equation of one of these planes is a constant multiplier of the other plane.

3.9 If $\mathbf{v}_1 \times \mathbf{v}_2 \neq 0$ and k_1 and k_2 are arbitrary constants not both zero, then $(k_1\mathbf{va}_1 + k_2\mathbf{va}_2) \cdot \mathbf{l} \cdot \mathbf{r} = 0$ is the equation of a pencil of planes whose common line of intersection is $\begin{cases} \mathbf{va}_1 \cdot \mathbf{l} \cdot \mathbf{r} = 0 \\ \mathbf{va}_2 \cdot \mathbf{l} \cdot \mathbf{r} = 0 \end{cases}$

This follows from the fact that $(k_1\mathbf{va}_1 + k_2\mathbf{va}_2) \cdot \mathbf{l} \cdot \mathbf{r} = 0$ becomes $k_1(a_1x + b_1y + c_1z + d_1) + k_2(a_2x + b_2y + c_2z + d_2) = 0$ the cartesian equation of a pencil of planes.

3.10 If $\mathbf{v}_1, \mathbf{v}_2$ and \mathbf{v}_3 are such that $\mathbf{v}_1 \times \mathbf{v}_2 = 0$ and $\mathbf{v}_2 \times \mathbf{v}_3 \neq 0$ then 1) $\mathbf{va}_1 \cdot \mathbf{l} \cdot \mathbf{r} = 0$ and 2) $\mathbf{va}_2 \cdot \mathbf{l} \cdot \mathbf{r} = 0$ are the equations for two parallel planes and 3) $\mathbf{va}_3 \cdot \mathbf{l} \cdot \mathbf{r} = 0$ is a third cutting the first two.

This follows from the fact that if the cross product of the constant vectors in the first two planes is zero, the coefficients of the x 's, y 's, and z 's in the cartesian equations of those planes are proportional; and if the cross product of the constant vectors in the second and third plane is not zero, the coefficients of the x 's, y 's and z 's in the cartesian equations of those planes are not proportional.

4. Application to Quadric Surfaces.

4.1 Equation of non-central quadric.

If Φ is uniplanar, $\mathbf{r} = xi + yj + zk$ is a variable vector, $\mathbf{v} = ai + bj + ck$ is any constant vector, and k is any scalar, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = \mathbf{v} \cdot \mathbf{l} \cdot \mathbf{r} + k$ is the equation of a non-central quadric.

This follows from the fact that $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = \mathbf{v} \cdot \mathbf{I} \cdot \mathbf{r} + k$ becomes $a_{11}x^2 + a_{22}y^2 = ax + by + cz + k$ a non-central quadric.²

4.2 Equation of Paraboloid.

If $\Phi = \pm \frac{\mathbf{ii}}{a^2} \pm \frac{\mathbf{jj}}{b^2}$, $\mathbf{v} = 2c\mathbf{k}$ and $c \neq 0$ then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 2c\mathbf{k} \cdot \mathbf{I} \cdot \mathbf{r}$ is the equation of a paraboloid.

If the signs of the two dyads of Φ are different and c is positive, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 2c\mathbf{k} \cdot \mathbf{I} \cdot \mathbf{r}$ is the equation of an hyperbolic paraboloid.

This follows from the fact that $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 2c\mathbf{k} \cdot \mathbf{I} \cdot \mathbf{r}$ becomes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 2cz$, the cartesian equation of a hyperbolic paraboloid.

If the signs of the two dyads of Φ are alike, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 2c\mathbf{k} \cdot \mathbf{I} \cdot \mathbf{r}$ is the equation of an elliptic paraboloid.

This follows from the fact that $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 2c\mathbf{k} \cdot \mathbf{I} \cdot \mathbf{r}$ becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \pm 2cz$, the cartesian equation of an elliptic paraboloid.

If $\Phi = \frac{\mathbf{ii}}{4a} + \frac{\mathbf{jj}}{4a}$, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = \mathbf{k} \cdot \mathbf{I} \cdot \mathbf{r}$ is the equation of a paraboloid of revolution.

This follows from the fact that $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = \mathbf{k} \cdot \mathbf{I} \cdot \mathbf{r}$ becomes $x^2 + y^2 = 4az$, the cartesian equation of a paraboloid of revolution.³

4.3 Equation of the polar plane to a paraboloid.

If \mathbf{s} be a point in the polar plane of the point determined by the vector \mathbf{a} , then $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 2c\mathbf{k} \cdot \mathbf{I} \cdot \frac{\mathbf{s} + \mathbf{a}}{2}$ is the equation of the polar plane to a paraboloid.

This follows from the following facts:

A line joining \mathbf{a} and \mathbf{s} is $\frac{y\mathbf{s} + x\mathbf{a}}{x + y}$. If this point is on the paraboloid, then

$$\frac{y\mathbf{s} + x\mathbf{a}}{x + y} \cdot \Phi \cdot \frac{y\mathbf{s} + x\mathbf{a}}{x + y} = 2c\mathbf{k} \cdot \mathbf{I} \cdot \frac{y\mathbf{s} + x\mathbf{a}}{x + y}.$$

This reduces to $(\mathbf{s} \cdot \Phi \cdot \mathbf{s} - 2c\mathbf{k} \cdot \mathbf{I} \cdot \mathbf{s})\left(\frac{y}{x}\right)^2 + [2\mathbf{s} \cdot \Phi \cdot \mathbf{a} - 2c\mathbf{k} \cdot \mathbf{I} \cdot (\mathbf{s} + \mathbf{a})]\frac{y}{x} + (\mathbf{a} \cdot \Phi \cdot \mathbf{a} - 2c\mathbf{k} \cdot \mathbf{I} \cdot \mathbf{a}) = 0$. The two values of the ratio y/x determined by this equation are equal in magnitude and opposite in sign, since \mathbf{s} is on the polar plane of \mathbf{a} . Consequently, the term in y/x vanishes. Thus $2\mathbf{s} \cdot \Phi \cdot \mathbf{a} - 2c\mathbf{k} \cdot \mathbf{I} \cdot (\mathbf{s} + \mathbf{a}) = 0$ reduces to $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 2c\mathbf{k} \cdot \mathbf{I} \cdot \frac{\mathbf{s} + \mathbf{a}}{2}$, the required equation.

4.4 Reciprocal relationship of poles and polars.

If \mathbf{b} is on the polar of \mathbf{a} , then \mathbf{a} is on the polar of \mathbf{b} .

² James Loyd Shawn, *The Dyadic Operator and Conics*, p. 22.

³ James Loyd Shawn, *The Dyadic Operator and Conics*, p. 22.

This follows from the following facts:

- 1) $\mathbf{b} \cdot \Phi \cdot \mathbf{a} = 2\mathbf{c}\mathbf{k} \cdot \mathbf{I} \cdot \frac{\mathbf{b} + \mathbf{a}}{2}$ by hypothesis.
- 2) $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 2\mathbf{c}\mathbf{k} \cdot \mathbf{I} \cdot \frac{\mathbf{s} + \mathbf{a}}{2}$ is the polar of \mathbf{a} .
- 3) $\mathbf{s}^1 \cdot \Phi \cdot \mathbf{b} = 2\mathbf{c}\mathbf{k} \cdot \mathbf{I} \cdot \frac{\mathbf{s}^1 + \mathbf{b}}{2}$ is the polar of \mathbf{b} .

Since 1) is true, it follows that $\mathbf{a} = \mathbf{s}^1$ satisfies 3).

4.5 Tangent plane to the paraboloid.

If \mathbf{a} is on the paraboloid, then the polar of the point \mathbf{a} is the tangent plane to the paraboloid at that point.

This follows from the following facts:

$$\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 2\mathbf{c}\mathbf{k} \cdot \mathbf{I} \cdot \mathbf{r} \text{ has for its total differential } 2\mathbf{d}\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 2\mathbf{c}\mathbf{k} \cdot \mathbf{I} \cdot \mathbf{d}\mathbf{r}.$$

The total differential of $\mathbf{s} \cdot \Phi \cdot \mathbf{r} = 2\mathbf{c}\mathbf{k} \cdot \mathbf{I} \cdot \frac{\mathbf{s} + \mathbf{r}}{2}$ is $\mathbf{d}\mathbf{s} \cdot \Phi \cdot \mathbf{r} = 2\mathbf{c}\mathbf{k} \cdot \mathbf{I} \cdot \frac{\mathbf{d}\mathbf{s}}{2}$; but this equation is satisfied by $\mathbf{d}\mathbf{s} = \mathbf{d}\mathbf{r}$.

4.6 Anti-self-conjugate dyadic.

If Φ is an anti-self-conjugate dyadic, the product of $\mathbf{r} \cdot \Phi \cdot \mathbf{r}$ is identically zero for all values of \mathbf{r} .

This follows from the following facts:

$\Phi = \frac{1}{2}(\Phi - \Phi^c)$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. $\Phi = a_{ii}\mathbf{i}\mathbf{i} + b_{jj}\mathbf{j}\mathbf{j} + c_{kk}\mathbf{k}\mathbf{k}$ can be expressed as $\Phi = \frac{1}{2}[(a_{ii}\mathbf{i}\mathbf{i} + b_{jj}\mathbf{j}\mathbf{j} + c_{kk}\mathbf{k}\mathbf{k}) - (a_{ii}\mathbf{i}\mathbf{i} + b_{jj}\mathbf{j}\mathbf{j} + c_{kk}\mathbf{k}\mathbf{k})]$. $\mathbf{r} \cdot \Phi \cdot \mathbf{r}$ gives $(ax^2\mathbf{i} \cdot \mathbf{i} + by^2\mathbf{j} \cdot \mathbf{j} + cz^2\mathbf{k} \cdot \mathbf{k}) - (ax^2\mathbf{i} \cdot \mathbf{i} + by^2\mathbf{j} \cdot \mathbf{j} + cz^2\mathbf{k} \cdot \mathbf{k}) \equiv 0$.

4.7 Locus of the mid-points of a system of parallel chords of a paraboloid.

If $\mathbf{s} = n\mathbf{i} + m\mathbf{k} + l\mathbf{a}$ be the mid-point of one of the chords of the system parallel to \mathbf{a} , then the locus of the mid-points of a system of parallel chords of a paraboloid is a plane parallel to the axis.

This follows from the following facts:

The equation of the paraboloid is $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 2\mathbf{c}\mathbf{k} \cdot \mathbf{I} \cdot \mathbf{r}$. $\mathbf{r} = \mathbf{s} + x\mathbf{a}$ is the equation of the secant line upon which the chord lies. Solving for points of intersection of secant and paraboloid, we have $(\mathbf{s} + x\mathbf{a}) \cdot \Phi \cdot (\mathbf{s} + x\mathbf{a}) = 2\mathbf{c}\mathbf{k} \cdot \mathbf{I} \cdot (\mathbf{s} + x\mathbf{a})$ which gives $x^2(\mathbf{a} \cdot \Phi \cdot \mathbf{a}) + x(2\mathbf{s} \cdot \Phi \cdot \mathbf{a} - 2\mathbf{c}\mathbf{k} \cdot \mathbf{a}) + \mathbf{s} \cdot \Phi \cdot \mathbf{s} - 2\mathbf{c}\mathbf{k} \cdot \mathbf{s} = 0$. Since \mathbf{s} bisects the chosen chord the two solutions for x are equal in magnitude and opposite in sign. Then $2\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 2\mathbf{c}\mathbf{k} \cdot \mathbf{a}$. Since $\mathbf{s} = n\mathbf{i} + m\mathbf{k} + l\mathbf{a}$; this is a plane parallel to the \mathbf{j} axis if l is constant. Substituting $n\mathbf{i} + m\mathbf{k} + l\mathbf{a}$ for \mathbf{s} in $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = \mathbf{c}\mathbf{k} \cdot \mathbf{a}$ gives $\frac{n\mathbf{i} \cdot \mathbf{a}}{a^2} + l\mathbf{a} \cdot \Phi \cdot \mathbf{a} = \mathbf{c}\mathbf{k} \cdot \mathbf{a}$. Construct \mathbf{i} perpendicular to \mathbf{a} .

it follows that $\mathbf{i} \cdot \mathbf{a} = 0$. Therefore $l\mathbf{a} \cdot \Phi \cdot \mathbf{a} = \mathbf{c}\mathbf{k} \cdot \mathbf{a}$ or $l = \frac{\mathbf{c}\mathbf{k} \cdot \mathbf{a}}{\mathbf{a} \cdot \Phi \cdot \mathbf{a}}$, a constant.

4.8 Generating a central quadric.

If $\mathbf{r}^1 = \Phi \cdot \mathbf{r}$ and \mathbf{r} is allowed to generate a sphere, then \mathbf{r}^1 generates a central quadric.

This follows from the following facts:

$\mathbf{r}^1 = \Phi \cdot \mathbf{r}$; $\mathbf{r} = \Phi^{-1} \cdot \mathbf{r}^1$; $\mathbf{r} \cdot \mathbf{r} = 1 = \mathbf{r}^1 \cdot (\Phi^{-1} \cdot \Phi^{-1}) \cdot \mathbf{r}^1 = \mathbf{r}^1 \cdot \psi \cdot \mathbf{r}^1$ where $\psi = (\Phi^{-1} \cdot \Phi^{-1}) = a_{ii} + b_{jj} + c_{kk}$. If $\mathbf{r}^1 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot$

$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 1$ reduces to $ax^2 + by^2 + cz^2 = 1$, a central quadric.

4.9 Quadric surfaces.

If $\Phi = \pm \frac{ii}{a^2} \pm \frac{jj}{b^2} \pm \frac{kk}{c^2}$ where any combinations of signs may occur, $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ and $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$ give rise to the central quadric surfaces; the real ellipsoid, the hyperboloid of one sheet, an hyperboloid of two sheets, the imaginary ellipsoid, and the real or imaginary cone.⁴

If all the signs of Φ are positive, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is the equation of a real ellipsoid.

This follows from the fact that $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, the cartesian equation of the real ellipsoid.

If one of the signs of Φ is negative, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is the equation of the hyperboloid of one sheet.

This follows from the fact that $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, the cartesian equation of the hyperboloid of one sheet.

If two of the signs of Φ are negative, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is the equation of an hyperboloid of two sheets.

This follows from the fact that $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ becomes $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, the cartesian equation of an hyperboloid of two sheets.

If all of the signs of Φ are negative, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is the equation of the imaginary ellipsoid.

This follows from the fact that $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$, the cartesian equation of the imaginary ellipsoid.

If all the signs of Φ are not alike, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$ is the equation of a real cone.

This follows from the fact that $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$ becomes $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, the cartesian equation of the real cone.

If all the signs of Φ are alike, then $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$ is the equation of an imaginary cone.

This follows from the fact that $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$ becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$, the cartesian equation of the imaginary cone.

4.10 Normal to the tangent plane to the central quadric.

If Φ is as in 4.9, then $\Phi \cdot \mathbf{r}$ is normal to the tangent plane to the central quadric.

This follows from the following facts:

The total differential of $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is $d\mathbf{r} \cdot \Phi \cdot \mathbf{r} + \mathbf{r} \cdot \Phi \cdot d\mathbf{r} = 0$; $2d\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$; $d\mathbf{r} \cdot (\Phi \cdot \mathbf{r}) = 0$; $d\mathbf{r}$ is along the tangent.

4.11 The magnitude of the normal to the central quadric.

⁴ James Loyd Shown, *The Dyadic Operator and Conics*, pp. 22-23.

If $N = \Phi \cdot \mathbf{r}$, the normal at \mathbf{r} , then $\Phi \cdot \mathbf{r}$ is equal to the reciprocal of the distance from the origin to the tangent plane at \mathbf{r} .

This follows from the following facts:

Since $N = \Phi \cdot \mathbf{r}$ and $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$, then $\mathbf{r} \cdot N = 1$ or $\mathbf{r} \cdot nN_0 = 1$, where N_0 is a unit vector along N . Let $\mathbf{r} \cdot N_0 = p$, the distance from the origin to the tangent plane at \mathbf{r} . Then $1/p = n$.

4.12 The radius vector of a sphere.

If \mathbf{p} be the vector from the center of the sphere perpendicular to the tangent plane, then the radius vector, \mathbf{r} , of a sphere is perpendicular to the tangent plane.

This follows from the fact that $\mathbf{r} \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{p}$ and since r can not vary in magnitude, $\mathbf{r} = \mathbf{p}$.

4.13 The equation of the polar plane with respect to the quadric.

If \mathbf{s} be a point in the polar plane of the point determined by the vector \mathbf{a} , then $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 1$ is the equation of the polar plane of the pole \mathbf{a} with respect to that quadric.

The proof of this theorem is like that for 4.3 of this section.

4.14 The locus of the mid-points of a system of parallel chords of any quadric.

If \mathbf{s} be a vector to the mid-points of a system of parallel chords parallel to \mathbf{a} , then $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 0$ is a plane which passes through the origin, the locus of mid-points of parallel chords.

This follows from the following facts:

$\mathbf{r} = \mathbf{s} + x\mathbf{a}$ and $(\mathbf{s} + x\mathbf{a}) \cdot \Phi \cdot (\mathbf{s} + x\mathbf{a}) = 1$ gives $\mathbf{s} \cdot \Phi \cdot \mathbf{s} + 2x(\mathbf{s} \cdot \Phi \cdot \mathbf{a}) + x^2(\mathbf{a} \cdot \Phi \cdot \mathbf{a}) = 1$, a quadratic in s . The two values of x determined by this equation are equal in magnitude and opposite in sign, therefore, $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 0$.

4.15 The polar planes of all points on a line through the center of a central quadric.

The polar planes of all points on a line through the center of a central quadric are parallel.

This follows from the following facts:

$\Phi \cdot t\mathbf{r}_0 = t(\Phi \cdot \mathbf{r}_0)$. $\mathbf{s} \cdot \Phi \cdot t\mathbf{r}_0 = 1$ gives planes perpendicular to $\Phi \cdot \mathbf{r}_0$ for all values of t , cutting $\Phi \cdot \mathbf{r}_0$ at a distance $1/t$ from the origin. Thus $\mathbf{s} \cdot \Phi \cdot t\mathbf{r}_0 = 1$ is a family of parallel planes, with parameter t .

4.16 Secant plane of the central quadric.

If \mathbf{s} is on the polar of \mathbf{a} , determined by the tangent cone of the conic of intersection of the secant plane with the central quadric; and if \mathbf{b} , \mathbf{c} , and \mathbf{d} are vectors to any point of intersection of the secant plane with the central quadric, then the secant plane is the polar of the point \mathbf{a} .

This follows from the following facts:

$\mathbf{r} \cdot \Phi \cdot (\mathbf{b} + \mathbf{c} + \mathbf{d}) = 1$ is a secant plane. \mathbf{a} is the point of intersection of the three tangent planes determined by the vectors \mathbf{b} , \mathbf{c} , and \mathbf{d} .

1) $\mathbf{a} \cdot \Phi \cdot \mathbf{d} = 1$ is the tangent plane at \mathbf{b} .

2) $\mathbf{a} \cdot \Phi \cdot \mathbf{c} = 1$ is the tangent plane at \mathbf{c} .

3) $\mathbf{a} \cdot \Phi \cdot \mathbf{d} = 1$ is the tangent plane at \mathbf{d} .

4) $\mathbf{s} \cdot \Phi \cdot \mathbf{a} = 1$ is the polar of the point \mathbf{a} .

\mathbf{b} , \mathbf{c} , and \mathbf{d} satisfy equation 4). Therefore $\mathbf{r} \cdot \Phi \cdot (\mathbf{b} + \mathbf{c} + \mathbf{d}) = 1$ is the polar of the point \mathbf{a} .

4.17 *The equation of the tangent cone at infinity of an hyperboloid.*

If $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is the equation of an hyperboloid, then the equation of the tangent cone at infinity is $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$.

This follows from the following facts:

$\mathbf{r} = m\mathbf{r}_0$, where \mathbf{r}_0 is a unit vector, is a point on the hyperboloid. The equation $\mathbf{s} \cdot \Phi \cdot m\mathbf{r}_0 = 1$ is that of the tangent plane of $m\mathbf{r}_0$. $\lim_{m \rightarrow \infty} (\mathbf{s} \cdot \Phi \cdot m\mathbf{r}_0 = 1)$ is the $\lim_{m \rightarrow \infty} \left(\mathbf{s} \cdot \Phi \cdot \mathbf{r}_0 = \frac{1}{m} \right)$, or $\mathbf{s} \cdot \Phi \cdot \mathbf{r}_0 = 0$. Since \mathbf{r}_0 has the direction of \mathbf{r} , and \mathbf{s} is perpendicular to $\Phi \cdot \mathbf{r}$, $\mathbf{s} \cdot \Phi \cdot \mathbf{r} = 0$, which is a plane through the origin. In this plane $\mathbf{s} = t\mathbf{r}$, hence $t\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$, or $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$, which is $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$.

4.18 *The two segments of a secant line parallel to \mathbf{a} intercepted between the cone $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$ and the hyperboloid $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$.*

If \mathbf{a} is a chord, then the two segments of a secant line parallel to \mathbf{a} intercepted between the cone $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$ and the hyperboloid $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ are equal.

This follows from the following facts:

Let \mathbf{b} be a vector from the origin to the mid-point of that part of \mathbf{a} which is the chord falling on the given secant. $\mathbf{r} = \mathbf{b} + x\mathbf{a}$ is the equation of the secant. $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is that of the hyperboloid. $(\mathbf{b} + x\mathbf{a}) \cdot \Phi \cdot (\mathbf{b} + x\mathbf{a}) = 1$ gives $\mathbf{b} \cdot \Phi \cdot \mathbf{b} + 2x\mathbf{b} \cdot \Phi \cdot \mathbf{a} + x^2\mathbf{a} \cdot \Phi \cdot \mathbf{a} = 1$ whose solutions give the intersections of the secant with the hyperboloid. Since the two values of x are equal and opposite in sign, $\mathbf{b} \cdot \Phi \cdot \mathbf{a} = 0$. The common solution of the secant, and the cone, $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 0$, are involved in $(\mathbf{b} + x'\mathbf{a}) \cdot \Phi \cdot (\mathbf{b} + x'\mathbf{a}) = 0$, i.e., $\mathbf{b} \cdot \Phi \cdot \mathbf{b} + 2x'\mathbf{b} \cdot \Phi \cdot \mathbf{a} + x'^2\mathbf{a} \cdot \Phi \cdot \mathbf{a} = 0$. But $\mathbf{b} \cdot \Phi \cdot \mathbf{a} = 0$, therefore the coefficient of $2x'$ vanishes, which means the two roots of this equation in x' are equal in magnitude and opposite in sign.

4.19 *A position vector describing an ellipsoid or hyperboloid.*

If $\mathbf{r}' = \Phi \cdot \mathbf{r}$, where $\mathbf{r} \cdot \Phi \cdot \mathbf{r} = 1$ is an ellipsoid or hyperboloid, \mathbf{r}' , as a position vector describes an ellipsoid or hyperboloid, respectively.

This follows from the following facts: $\mathbf{r}' = \Phi \cdot \mathbf{r}$, then $\mathbf{r} = \mathbf{r}' \cdot \Phi^{-1}$ and $\mathbf{r}' \cdot \Phi^{-1} \cdot \Phi \cdot \mathbf{r}' \cdot \Phi^{-1} = 1$; but $\mathbf{r}' \cdot \Phi^{-1} \cdot \Phi \cdot \Phi^{-1} \cdot \mathbf{r}' = 1$ gives $\mathbf{r}' \cdot \Phi^{-1} \cdot \mathbf{r}' = 1$.

THE SPECTROGRAPH AND ITS USE IN SPECTROCHEMICAL ANALYSIS

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I. GENERAL SPECTROSCOPY

*A. History.*¹ The spectrum in its natural form—the rainbow—was not connected with the simpler instances of refraction, even by the most advanced of the older civilizations. Until Newton's time, the scientists had only vague ideas concerning the nature of color. Some even believed that white light was changed into different colors within the prism or refracting body, and that color was a mixture of light and darkness.

The most advanced step in bridging the gap between these strange, inexact notions and our modern theory of the relations between light and color on the one hand, and atomic and molecular structure on the other, was taken by Sir Isaac Newton in 1672. At that time his experiments, which were simple but fundamental, were described in the *Transactions of the Royal Society*. He allowed sunlight, entering a round hole in a shutter, to pass through a glass prism and fall on a screen. He called the resulting phenomenon a spectrum and recognized it as a series of colored images of the entrance hole. From this experiment he concluded that a ray of sunlight is a combination of a number of rays of various colors, each of which has a different index of refraction when passed through a glass prism.

Newton understood that the sorting of rays could be carried further and the purity of the spectrum improved by the use of a lens. This he usually placed before the prism and used it to image the entrance hole or slit on the screen. He thus had a spectroscope in nearly modern form, and with it he formed solar spectra as long as ten inches. Newton realized that different colors must have different indices of refraction, and he made measurements of the indices of various substances. His clarification of the nature of color laid the foundation for all later work. However, he speculated that all colors of flames were of the same nature and thus he delayed the practical applications of spectroscopy by missing the one fundamental principle—the fact that emitted light is characteristic of the atoms or molecules that produces it. Nearly 200 years elapsed before this principle was thoroughly established.

* The author's thanks are due to Drs. W. A. Everhart and to L. E. Smith, without whose aid, both with apparatus and suggested procedure, this investigation would not have been concluded.

¹ The material in the succeeding section comes mainly from Ralph A. Sawyer, *Experimental Spectroscopy*, (New York, 1944), Chapter I.

Newton's spectrum was considerably confused, because he used a round hole as a slit and the images of different colors overlapped one another. It was not until 1802 that W. H. Wollaston made the improvement of using a narrow slit to eliminate all unequal overlapping of its colored images. On further experiment, he observed the dark lines which interrupt the continuous spectrum of the sun in a direction parallel to the slit. He did not investigate these lines, so their explanation was left to Fraunhofer, by whose name these lines are known today.

Joseph Fraunhofer was born near Munich in 1787, and became a celebrated optician. In 1814, seeking to improve the method of defining the color of light used in measuring the index of refraction of glasses, he made a detailed examination of the spectrum of sunlight. Using a telescope placed behind the prism, he was able to make exact measurements of angles. From his experiments he found the spectrum crossed with "an almost countless number of strong and weak vertical lines."² Fraunhofer could not explain these lines, except that they assumed fixed positions in the solar spectrum, but he made a map of about 700 of them and assigned to eight of the more prominent ones the letters A to H, by which they are still known. These lines gave optical science the first definite standards for the comparison of the dispersion of different glasses, and the foundation stone of the modern science of spectroscopy was laid.

The second contribution of Fraunhofer was connected with his investigation into the possibilities of measuring the actual length of the waves of light. He studied the behavior of fine slits and observed their diffraction patterns. Following the first discovery of diffraction by Grimaldi in 1665, Thomas Young had explained the principles of interference. Fraunhofer succeeded in developing the theory of interference of light so far as to invent and construct the first diffraction grating, by means of which he made several accurate measurements of the wave length of the D line of sodium. These measurements, and others on several other Fraunhofer lines, were the first wave-length measurements of spectral lines. (See page 114.)

Spectroscopy was put on a scientific basis when the fact that each atom and molecule has its own characteristic spectrum was recognized. Fraunhofer moved toward this recognition when he noticed that the yellow lines in a flame were double and seemed identical in wave length with the two dark D lines in the solar spectrum. This, however, was not fully explained for some time, because sodium, whose spectrum has the D lines, is one of the most common contaminations. For many years investigators were on the verge of discovering this basic principle.

J. F. W. Herschel made extensive studies of the flame spectra of salts, and he came near to the truth by recognizing that some substances give definite spectra, but he failed to see that one spectrum can belong to only one substance.

Many others during the first half of the nineteenth century observed and described various flame and arc spectra. Among them are Sir Charles Wheatstone, the English physicist, who worked on metallic sparks; Angstrom, the Swed-

² As quoted in Sawyer, p. 5, from J. S. Ames, ed., *Prismatic and Diffraction Spectra*, (New York, 1898).

ish physicist, and Alter, the American physician. These men all noted the characteristic differences in different spectra, but they failed to recognize the full significance of the discovery. Angstrom also explained gaseous and absorption spectra. In 1848, the French physicist, Foucault, observed absorption of the D lines of sodium when a stronger light of an arc was placed behind a sodium flame. The answer to many questions waited for the results of Kirchhoff's researches.

While Professor of Physics at Heidelberg, Kirchhoff announced his law and its mathematical proof in 1859, and pointed out its applications in two papers before the Berlin Academy of Science. Kirchhoff's law states: "The relation between the powers of emission and the powers of absorption for rays of the same wave length is constant for all bodies at the same temperature."³ Therefore, a gas which radiates a line spectrum must absorb the lines which it radiates at the same temperature. He explained the dark Fraunhofer lines in the solar spectrum as caused by the absorption by the elements in the cooler sun's atmosphere of the continuous spectrum emitted by the hot interior of the sun.

Kirchhoff then worked for the chemical analysis of the sun's atmosphere in conjunction with Robert Bunsen, Professor of Chemistry at Heidelberg. These two men also applied themselves to the problem of chemical analysis with the spectrometer. Their demonstration of the power of the spectrometer gave a new impetus to modern spectroscopy, and great popular interest was aroused.

The weakness of the great volume of work which followed was a common lack of accuracy and of suitable standards of wavelength. The first useful standards were provided by Angstrom, in his research on the solar spectrum. He measured and charted the wave lengths of about one thousand of the Fraunhofer lines. This map was called by Angstrom the *Normal Solar Spectrum*, and was published in 1868. His research was carried out with the help of three glass gratings, very carefully ruled. The measurements covered the region of the visible spectrum (about 4000-7500 Å), and the wave-lengths were expressed in 100^{-8} cm. That unit has been used ever since in wave-length determinations as the Angstrom unit.

Angstrom's values were found to be inaccurate because of his error in measuring the grating spaces of his glass gratings. His pupil, Thalen, in 1884, discovered this fact and set himself to correcting the measurements. He also extended the work by determining the wave-lengths of the spark spectra of all the metals then known (44) by means of a prism spectroscope.

The publication of Angstrom's map marks a definite stage in the development of spectroscopy. For the first time, a standard of reference, placed upon a physical basis, was on record. The determinations of wave-lengths of lines of unknown spectra was now made possible by a direct comparison between the unknown spectrum and that of the sun.

About 1870, with the development of the dry gelatine photographic plate, photographic methods were found to be more accurate than the previous visual methods. Numerous attempts were made in the next few years to provide better standard wave-lengths. Henry Rowland, professor at Johns Hopkins University, A. A. Michelson, professor at the University of Chicago, Fabry, and Perot at Paris were working on the table of wave-lengths.

³ Sawyer, p. 8.

In 1904, the newly formed International Union for Cooperation in Solar Research met in St. Louis to discuss the question of suitable standards. The following year the meeting was held in Oxford, and it was agreed "to adopt a suitable spectroscopic line as the primary standard and to determine secondary standards by interferometric comparison with this primary standard."⁴

The first important extension of the ultraviolet spectrum was made in England by G. G. Stokes, who, in 1862, discovered the transparency of quartz in this region, and observed the arc and spark spectra of numerous metals. The first wave-length determinations in this region were made by Mascart in France in 1863.

The ultraviolet spectrum was extended by several men who modified the usual methods. Victor Schumann succeeded in photographing the region to 1200 Å by making photographic plates almost free from gelatin, by substituting for quartz prisms and lenses, optics made of fluorite, and by enclosing the whole equipment in an evacuated chamber to eliminate the absorption of the oxygen. A great extension of the ultraviolet region was achieved in 1906, by Theodore Lyman, who employed the concave grating instead of the fluorite optics. Later improvements in light sources, grating ruling, and mounting by Millikan, Bowen, Sawyer, and Siegbahn have progressively reduced the limit of the ultraviolet region to below 30 Å.

Rowland's measurements in the infrared region reached a maximum at 7715 Å the limit of sensitivity of the usual red-sensitive photographic emulsion. Commercial photographic plates are now sensitive to about 12,500 Å. The big extensions of the infrared region of the spectrum have been made by non-photographic, physical devices which are sensitive to radiant energy. Professor S. P. Langley of Washington, D. C., in 1881, devised the bolometer, a very sensitive electrical resistance thermometer, by means of which it is possible to extend the solar spectrum to 180,000 Å. Tremendous advances were made in Germany by Rubens and Paschen from 1892 on. Together with their pupils, they were very active in all phases of infrared research. As a result of research with various techniques and equipment, the radiation spectrum has been investigated far into the infrared, to wave-lengths of about 3,000,000 Å.

The whole period from 1860 almost to the present has been rich in fundamental discoveries, each opening new fields of spectrographic research and leading to the enlargement and refinement of our understanding of the atom and of its mathematical theory. The spectrograph has been found useful in other fields of research, such as astronomy, biology, physics, and chemistry. In applied science, the use of the spectrograph has been rapidly increasing. Spectrochemical analysis has been a means of obtaining rapid and precise chemical analyses.

B. Principles. A spectrograph is an instrument which disperses the different wave-lengths of light to provide a means of qualitative or quantitative study of the spectrum thus formed. In general, the spectrograph consists of: (1) an entrance aperture, usually a rectangular slit, whose images are the spectral lines

⁴ Sawyer, p. 11.

which are photographed; (2) a dispersing device which gives to different wave-lengths passing through it different emergent angles; (3) an optical system of mirrors or lenses to bring the different wave-lengths to focus.

The light enters the spectrograph by means of a rectangular slit. Generally speaking, the slit is formed of two metal jaws operated by some mechanism which will cause either one or both jaws to move perpendicularly to the edges. The essential requirements for a good slit are sharp, parallel, and straight edges which are readily adjustable. The slit should be protected from dust and corrosion to insure sharp edges.

An important characteristic of any spectrograph is its dispersion, or power to spread different wave-lengths out, so that they emerge from the dispersing system at different angles and are focused in different positions in the focal plane of the spectrograph. The difference in angles for different wave-lengths is measured by the angular dispersion, defined as $\Delta\theta/\Delta\lambda$, where $\Delta\theta$ is the difference in emergent angles corresponding to a difference in wave-length, $\Delta\lambda$. The angular dispersion may be expressed in radians or degrees per Angstrom unit. It depends on the refracting angle of the prism and nature of the medium used in the case of a prism spectrograph. In a diffraction grating type of spectrograph, the dispersion depends on the space between the lines in the grating and the angle of refraction. The linear dispersion, or separation of different wave-lengths in the focal plane, is expressed as $\Delta x/\Delta\lambda$, where Δx is the distance between two spectral lines which differ in wave-length by $\Delta\lambda$. The linear dispersion is expressed in centimeters per Angstrom. To alter dispersion the index of refraction of the prism or the angle of the prism must be changed.⁵

The theoretical resolving power is defined as "the ratio of the wave-length observed to the smallest difference between two wave-lengths which can just be resolved or distinguished as two separate lines, $R = \lambda/\Delta\lambda$."⁶ The resolving power is dependent only on the dispersion and the base of the prism used. This value for R is the *theoretical* resolving power and, therefore, it is computed for an infinitely narrow slit and for an infinitely narrow line. Neither of these conditions can hold in practice, so the actual resolving power is always less than the theoretical power. To alter resolving power the base of the prism must be changed, at the same time utilizing its full aperture. Actual resolution is affected by many factors however, such as the relative intensities of the two adjacent spectral lines, their form or structure, the width of the slit, the method of illumination of the slit, and the perfection of the optics and their adjustment.⁷

The optical system of all instruments consists of a collimator lens and a telescope or camera lens. The former collects the rays coming through the slit and transmits them as a parallel beam onto the prism face. The latter collects the rays leaving the last prism face and brings them to a focus at the focal plane. The slit is placed in the focal plane of the collimator, and the photographic plate is placed in the focal plane of the telescope lens. Between the two lenses, in the

⁵ *Spectrographic Equipment* (Rochester), Cat. D-20, p. 4.

⁶ Sawyer, p. 30.

⁷ Sawyer, p. 33.

region where the beam is parallel, the dispersing system is placed. To obtain the maximum illumination from the light source, the collimator lens must be filled with light. If the source, or the part of it which is to be used, is too small to fill the collimator with light, the use of a condensing lens to form an image of the source at or near the slit will increase the illumination of the spectrum. A condensing lens serves several purposes. In the first place, it provides an index for the proper location of the light source. Secondly, the condensing lens insures the proper illumination of the whole slit so that the lines in the spectrum are sharply defined. Most important of all, it insures the complete filling of the collimator lens with light, thus utilizing the full resolving power to insure clean separation of closely adjacent lines, and making the most effective use of the available light.⁸

Light sources are classified according to the spectrum they emit, as continuous or discontinuous. A continuous spectrum is characterized by a generally uninterrupted range of wave-lengths over a considerable region and by the absence of sharp lines or bands. In the visible region this spectrum appears as an unbroken series of colors. Discontinuous sources show a number of sharp, narrow, "bright lines" in their spectrum. The individual atoms or molecules in the light source, which are usually in the vapor or gaseous state, emit radiations which can be converted to a spectrum.

Continuous spectra are characteristic of incandescent solids and liquids, such as lamp filaments, of certain high-pressure discharges, and of certain gaseous discharges. Discontinuous spectra are radiated from atoms or molecules in a gaseous or vapor state which have stored up excess energy in some manner and then radiate it as luminous energy. Sources differ in the method by which energy is supplied to the atoms and in the amount which can be absorbed before radiation. Atoms or molecules may receive energy by absorption or radiation, as kinetic energy from collisions with electrons or atoms, by taking energy excitation from other atoms or molecules which have absorbed energy, or by thermal excitation.⁹

To understand the theory of absorption and emission, it is necessary to consider the structure of the atom. According to the modern atomic theory, the atom is composed of a positively charged nucleus surrounded with electrons that are revolving in orbits. These orbits may be considered as representing certain energy levels of the electrons in the atom. The addition of energy to an atom will cause the electrons to move out into larger orbits in order to absorb the added energy. On the other hand, when an electron drops back into one of the orbits of an atom, it should emit energy, since energy is required to drive the electron from the atom. The fundamental concept in interpreting spectral lines is the absorption and emission of energy due to the movement of the electron from orbits, or energy levels, located at different distances from the nucleus. The spectrum of light due to energy emitted by electrons as they drop to smaller orbits in the atom is called an emission spectrum. The absorption spectrum is

⁸ *Spectrographic Equipment*, p. 7.

⁹ Sawyer, p. 18.

produced by absorption of energy, causing the electrons to move out to larger orbits. The solar spectrum, which shows dark Fraunhofer lines, is an absorption spectrum. It is due to the absorption by the gases in the sun's atmosphere of part of the energy emitted, which has the same frequency as that through which the light is passing.¹⁰

C. Prism Spectroscopy. The earliest spectroscopes and spectrographs were prism instruments. Newton was the first to use the prism spectroscope and improvements on his apparatus were made by Fraunhofer, Babinet, Kirchhoff, and Bunsen.¹¹ The modern spectrograph now consists of a slit, a collimator lens, a prism or prisms, a camera lens, and a photographic plate.

As stated above, the slit is a narrow, rectangular aperture which admits light to the spectrograph. The images of the slit formed by the spectrograph after the light has been dispersed are the spectral lines. It is essential, therefore, that the edges of the slit be sharp, straight, and parallel, in order to give a clean image. Adjustable slits are preferred for research work and for instruments of high dispersion, since the proper slit width for best results depends on the wave length used.¹² The slit should be protected from scratching and dust in order to keep the edges sharp.

The function of the collimator lens is to render parallel the beam of light coming from the light source. For collimator lenses there are three chief image defects: chromatic aberration, spherical aberration, and coma. Chromatic aberration was discovered by Newton, and arises from the fact that transparent media have different indices of refraction for rays of different wave-lengths. This may be overcome by using in a lens two materials with different dispersions or rates of changes of index of refraction with wave length. Tilting of the photographic plate will also reduce chromatic aberration. Spherical aberration is the failure of a lens to bring to a focus, at the same point, the rays which pass through zones at different distances from the axis. The results of spherical aberration will be a lack of parallelism of the light when it falls on the prism face, and the spectral line will not be sharp. This defect may be corrected in a single lens by altering the lens shape from a simple spherical section in such a way as to give all zones of the lens the same focal length. Coma is the spherical aberration of rays which pass through the lens obliquely. It results from the failure of the lens to form a sharp image of objects somewhat off the axis of the lens. Coma is reduced by the proper choice of curvature in a single lens.¹³

The dispersion of light in the prism spectrograph is due to the inherent characteristic of prisms—the refraction of light. Another image defect is introduced by the passage of light through a prism. The result of this defect—astigmatism—is that a point image is not formed of a point source; rather, two image-lines are

¹⁰ Evans, William L., Garrett, A. B., and Quill, L. L., *Semimicro Qualitative Analysis* (Boston, 1942), p. 210.

¹¹ Sawyer, p. 47.

¹² Sawyer, p. 48.

¹³ Sawyer, p. 53.

found at different distances from the optical system. Astigmatism in a prism is at a minimum if the following precautions are taken:

1. The prism is traversed by parallel light.
2. The slit is parallel to the prism edge.
3. The light rays are parallel to a section perpendicular to the edge.
4. The rays pass through the prism symmetrically, that is, with equal refraction at each surface. The rays then pass at minimum deviation."¹⁴

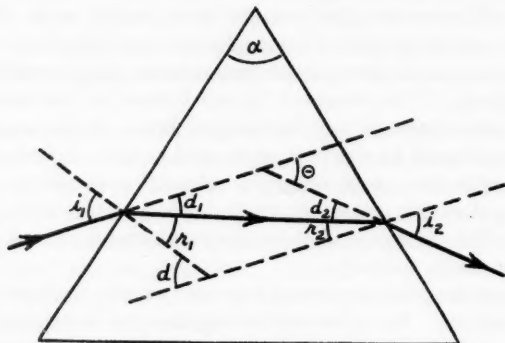


FIG. 1¹⁵

The position of minimum deviation, therefore, gives the most favorable image formation. When $i_1 = i_2$ and $r_1 = r_2$, the deviation, θ , is less than that for any other angle of incidence. The ray is then called the ray of minimum deviation.

The camera or telescope lens has to bring to a focus the beams of light of various wave lengths emerging from the prism so as to form images of the slit. This lens has to form an extended image of a very distant source, and consequently, new image defects appear—astigmatism and curvature of the image. Astigmatism, as stated above, arises from the aberration introduced in a beam by oblique passage through a lens. This can be removed by bringing the horizontal and vertical image-line surfaces together.¹⁶ Optical flare results when reflected light from the lens surface is brought to focus in such a way as to throw stray light on the photographic plate.

For any spectrograph, the factors which limit the definition of spectral lines at the two ends of the wave-length range may be different. With a prism instrument, the dispersion decreases logarithmically toward the longer wave lengths. Therefore, the resolving power at the red end of the spectrum is limited.

Mirrors may be substituted for collimator and camera lenses in spectrographs with marked advantages in many cases. In the extreme infrared, the use of mirrors is made necessary by the fact that ordinary optical material is not transparent to these radiations. In the visible and ultraviolet regions, mirrors are usually cheaper than lenses. Since all wave lengths are reflected alike in a mirror,

¹⁴ Ibid., p. 55.

¹⁵ Sawyer, p. 55.

¹⁶ Ibid., p. 77.

chromatic aberration is absent. However, coma and astigmatism cannot be easily corrected in a mirror figure.¹⁷

D. Grating Spectroscopy. The diffraction grating was discovered by Fraunhofer, and improved by Rowland and Michelson.¹⁸ This dispersing device has played an important role in spectroscopy and is replacing the prism in many applications. It is indispensable in the extreme ultraviolet and infrared regions, where no transparent media for prisms are known. In the visible region, the diffraction grating attains much greater dispersion and resolving power than the prism.

A diffraction grating is made up of a series of narrow slits, which are parallel, equidistant, and close together. The light is allowed to pass through these slits in the transmission grating. In the reflection grating, the light is reflected from close rulings on the mirrored surface, and these rulings act as individual slits.

The theory of the diffraction grating has been developed by Rowland. He found that the image of a slit formed by light which has passed through a series of apertures is a diffraction pattern. If the width is as small as the magnitude of a wave length λ , the maximum may become so wide as to cover the whole field of view. Because of interference new maxima and minima are found in this wide central field. With many slits, these maxima become narrow compared to the distances between them, and it is these maxima which constitute the spectra of the grating. The images of different wave lengths will be formed at different angles of diffraction, giving spectra of different orders.¹⁹

Like the prism, the plane grating requires the use of lenses or mirrors to collimate and to focus the incident and the emergent light. However, a concave mirror which is ruled can be used to focus the spectra formed by the grating. Rowland discovered and announced this in 1882, and in the next year he gave his theory of image formation of the grating.²⁰

E. The Photographic Process. The use of photography in spectrographic work has several advantages. Weak radiations can be recorded by longer exposures, and the plates can be retained as permanent records for reference. Also the plates can be studied at once or at any future time. Spectroscopes have none of these characteristics because of its inherent difference from the spectrograph—the visual means of observing the spectrum.

The photographic procedure is essentially the same as that used in ordinary photography. The formation of a photographic image depends on the photosensitivity of the silver halides. The action of light radiations on silver halide crystals has a latent effect which permits metallic silver to be deposited from the compound where it has been exposed to light. Reducing agents act as developers, leaving the metallic silver in the form of the image. The unreduced silver halide is removed by the aid of a solvent or "fixer", usually sodium thiosulfate or "hypo".²¹

¹⁷ Sawyer, p. 81.

¹⁸ Baly, E. C. C., *Spectroscopy* (London, 1905), p. 166-174.

¹⁹ Sawyer, p. 122.

²⁰ Ibid., p. 126.

²¹ Sawyer, p. 183.

F. Spectrochemical Analysis. "The primary standard of wave length is the wave length of the red cadmium line expressed in terms of the international meter bar. Direct comparison of other wave lengths with the red cadmium line, as the primary standard, is made by interferometric methods." This process is not adaptable to ordinary needs, so the iron lines have been chosen as secondary standards.²² These are distributed throughout the spectrum and are convenient to be used for the determination by interpolation of wave lengths of other elements.

In preparing the specimen to be studied, no elaborate chemical separations are required. If the sample is metallic or electrically conducting, it can be used directly as the electrodes. The sample will volatilize at the separated ends when the current flows, causing the emission of radiations from the ionized sample. If a powdered sample is to be studied, it can be placed in a bored cavity in the lower carbon or silver electrode. Solutions are analyzed by allowing the solution to be dried on the tips of the electrodes. No matter what form the sample may be, only a small amount is necessary for the analysis. This is introduced between the electrodes, which are sources of high temperature, hot enough to volatilize it and convert it into a luminous vapor. The light from this vapor is brought into the spectrograph which splits it up into its component, individual radiations. These appear as sharp lines, each characteristic of some one of the elements contained in the sample.

To interpret spectrograms qualitatively, the positions of the lines on the plate must be measured as accurately as possible. The identification of the lines can be made by the measurement of the wave lengths of the lines, or by comparison with the spectra of known elements, photographed adjacent to the spectrum to be studied. Complete measurement of all the lines in a spectrum is rarely needed in qualitative chemical analysis. It is usually only necessary to identify a few of the strongest and most persistent lines of each element to establish definitely its presence or absence. The iron spectrum is most commonly used as a standard comparison, since these lines are known to an extremely high degree of precision. Wave-length tables are available in which the iron lines are shown with the persistent lines of other elements, and by interpolation the wave length of the unknown line can be found. The possibility of interfering lines of other elements must be considered in identifying an individual line. By locating other persistent lines or groupings the positive identification can be reached.

In most cases, the major constituent is known or readily determined by inspection of the strongest lines or bands. Most of the other lines belonging to this constituent can then be identified by the use of the tabulations of spectra and a consideration of their relative intensities under similar electrical conditions. For a rapid identification of all the lines, spectra of pure samples of the major constituents may be photographed and compared.

The use of a large-scale calibration curve prepared from accurate measurements of a standard spectrum is a great convenience. From such a curve the wave

²² Sawyer, p. 205.

length can be read directly after measuring, by means of a magnifier with a ruled scale, the distance from the line studied to a line of known wave length.

By emission methods, all of the metallic elements may be identified; also, some nonmetallic elements, rare earth elements, and gaseous elements in discharge tubes can be observed. In spectrochemical work, it is not necessary for the elements to be present in uncombined form. The temperature of any of the ordinary sources is sufficient to dissociate any molecules and to excite the resulting atoms. This is one of the weaknesses of emission spectrochemistry, because little information of the state of combination of atoms in the source material is available.²³

The basis for quantitative determinations is the fact that the intensities of the spectral lines of an element are directly proportional to the concentration of that element in the sample. Under a given set of exposure conditions, the more of a given element present in a sample, the more numerous and the stronger will be its spectral lines. Quantitative work involves the measurement of the intensities of these lines, in comparison with standards of known and controllable intensity.

In comparing intensities, differences caused by variations in photographic material and development are eliminated by photographing the spectrum of the sample on the same plate with those of a series of standard samples. Visual methods are used to determine what standard concentration compares favorably with the sample spectrum.

More accurate results can be obtained by the "Internal Standard Method", which was developed by Gerlach and Schweitzer.²⁴ The comparison is not made with lines of the same element in separate standard spectra. Instead, the lines of the constituent to be determined are compared with lines of another element within the same spectrum. The reference standard must be present in constant or controllable amounts in the sample. Under absolutely constant conditions of excitation and photography, these reference lines will, therefore, have definite constant intensities.

Preliminary tests are made with the standard sample and with varying known amounts of the constituents to be determined. Calibrating a series of such tests will enable one to plot the intensities of the reference standard against the corresponding amount of the constituent to be determined later. Such a graph is often called a "working" curve, or a "gamma" curve, and is usually plotted on logarithmic scales. The curve will generally be a straight line from which the concentration in any similar sample may be obtained.

Comparisons of the intensities of lines is made on the basis of the relative opacity of the lines on the photographic plate. This relative blackening can be converted into light intensity ratios and graphed in these units. For accurate measurements of intensity a densitometer, or photometer, is used. In this instrument a light is directed on a single line and the transmitted portion of the light is directed onto a photoelectric cell. This is connected with a galvanometer which registers the amount of light transmitted by the spectral line.

²³ Sawyer, p. 298.

²⁴ *Spectrographic Equipment*, p. 10.

For quantitative analysis of a constituent known to be present in a concentration of more than five per cent, ordinary chemical methods of analysis are more desirable. Spectrochemical analysis is particularly valuable when a constituent is present in small amounts. Traces of an element can be detected to 0.001% concentrations easily, and analyses have been made to as low as 0.00001% in biological materials.²⁵ Thus this method of analysis finds many practical applications.

G. Practical Applications. Because of the extreme accuracy and rapid results of spectrochemical analysis, the spectrograph has been applied in many fields. In pure research it is an invaluable instrument. Astronomy has been advanced by the investigation of the composition of the solar bodies and their atmospheres. In physics, many problems involving discharge phenomena and reactions and elementary processes in electrical discharges are studied with the aid of the spectrograph. The study and analysis of the spectra of polyatomic molecules have been possible, as well as the investigation of photosynthesis and chemical reactions.²⁶

Besides emission spectroscopy, there is an increasing use of absorption spectra for the detection and study of organic and inorganic compounds. The absorption bands are more difficult to identify than ordinary emission lines. The radiations of the light emitted by an incandescent gas depends upon the vibrations of its constituent atoms. Therefore a study of light emitted shows its chemical composition. The measurement of the absorption of carbon dioxide in the infrared has been a means of studying animal and cellular respiration. The determination of Vitamin A was made possible by ultraviolet absorption measurements, as well as the positive identification of pepsin. Spectroscopic investigation has been of great assistance in the study of hemoglobin and blood analysis.²⁷

Perhaps one of the chief industrial applications of the spectrograph is in the iron and steel plants. The portions of the spectrum that are significant in alloys are analyzed qualitatively and quantitatively. The advantages of this method over the ordinary chemical methods are the permanent records which can be kept, detection of traces of impurities, and the rapid and accurate results. In the tin industry, also, the spectrograph has been used for analyses of alloys. The synthetic rubber industry and the high octane gasoline manufacturers are employing methods involving infrared analysis.²⁸

A recent application of spectrochemical analysis is in the field of scientific crime detection. The Federal Bureau of Investigation can now make analyses of minute quantities of paint, metals, mud, fibers, stains, etc., which are valuable in obtaining incriminating evidence.²⁹

²⁵ Brode, p. 126.

²⁶ Sawyer, p. 6.

²⁷ Melson, Everett W., "Spectroscopy: Its Applications," *Current Science*, 1936, V. IV, p. 517-519.

²⁸ Straat, Harold W., "Recent Applications of Infrared," *The Educational Focus* (Rochester, 1945), V. XVI, p. 15.

²⁹ Hoover, John E., "Optics in Scientific Crime Detection," *The Educational Focus*, V. XVII, p. 5.

Thus we can foresee the widening fields of application of the spectrograph, and the increasing use of this instrument in fields still unknown.

II. EXPERIMENTAL RESULTS

A. *Use of the Small Littrow Spectrograph.* 1. Apparatus.³⁰ The small Littrow type of spectrograph is a crystal quartz prism type with moderately high dispersion, but no great resolving power. The design of the optical system makes it "auto-collimating." This is a term which describes the double use of the lens, which acts both as the collimator and as the camera lens. Auto-collimation is possible because of the mirrored back of the prism in which the collimated light from the lens enters the prism and is reflected to the lens, after being dispersed, which focuses the rays on to the photographic plate by the same

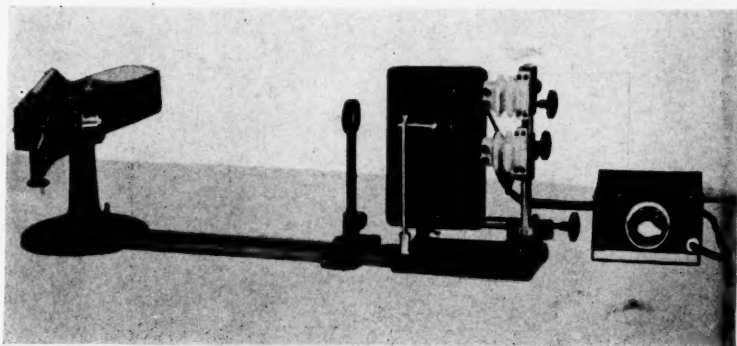


FIG. 2.³¹ Small Littrow Spectrograph complete with accessory equipment set up for use. On the right is the Arc and Spark Stand with Rheostat in the background. In the center stands the Condenser which focuses the image of the light source upon the slit in the Spectrograph.

lens which collimates the light. This design permits much greater focal length and linear dispersion for its overall length.

The photographic plate which is used in this instrument is seven inches in length, and therefore the linear dispersion is about 150 millimeters. The wavelength range covered is from 2100 to 7000 Å, and the individual spectral lines on the plate are three millimeters high. Once the lens and prisms are adjusted, they need not be changed, since the entire range is covered in one spectrogram.

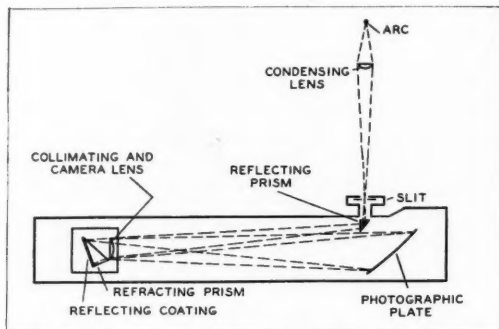
The slit in this instrument is unusual, in that four standard slit widths are provided—two, five, ten, and twenty microns. These have been cut in a silver and copper coating on a single quartz slide, which is then properly protected. The advantages of this slit include permanent stability, parallel edges, and ease of cleaning, since the metal coating is on the inner surface of the quartz plate

³⁰ The information concerning the construction of the instrument comes mainly from *Spectrographic Equipment*, p. 14-15, 22-23.

³¹ *Spectrographic Equipment*, p. 14.

where dust cannot enter. The four slits can be interchanged quickly and accurately, a spring stop holding it at any one of the four positions which are marked on the outer guides.

The shutter is operated by a rod located just behind the slit. It is of the simple, single-vane type, and may be locked in the open position when long exposures are desired. The plate holder is placed in position at the end of the focal length. It is constructed so that the plate is held firmly in place and pressed against curved edges so as to meet the focal curve of the lens, thus insuring good focus along the whole plate. The plate holder, when in position, is moved vertically by a crank located beneath its center. Thus a series of spectra can be photographed successively, one above the other, on the same plate. A millimeter reference scale can be printed on the plate adjacent to the spectrum. It is a photographic scale on thin glass, 150 millimeters in length. This scale may be placed in contact with the emulsion surface of the photographic plate by

FIG. 3³²

turning up the movable lever behind the plate holder. When the button on the base of the spectrograph is depressed for one or two seconds, the scale is illuminated by a blue-filtered lamp.

The other equipment includes a quartz condensing lens on a stand which is adjustable in height and lateral position, and an arc and spark stand. The electrodes are held in position by means of clamps which are vertically adjustable by rack and pinion mounts. Lateral adjustment of the electrodes is made by rotating the main single upright rod by means of a knob and screw. On the arc and spark stand is a large plate of blue plastic material, which serves as an eye shield. To regulate the height and separation of the electrode gap, an index pointer is supplied. All of these parts are placed in a dove-tail slide, so they can be locked in lateral adjustment.

Direct current is necessary for the arc; 220 volts is the preferred voltage. To maintain an even electrical pressure, a rheostat is used. It provides a regulator switch for currents of 2.25, 4.50, and 9.00 amperes.

³² *Spectrographic Equipment*, p. 9.

For studying the spectrograms, a spectrum plate viewing box is used. It is built in pyramidal form with a light bulb at the apex. There are four observation windows which are glazed with white milk glass to diffuse the light evenly. The plate is supported over each window by means of a shallow metal slot. For the measurement of the positions of the spectrum lines with reference to other lines in the same or adjacent spectra, a spectrum measuring magnifier is a great convenience, giving a magnification of 6.5 times. On the under side, it is ruled to give a scale twenty millimeters long, in 0.1 millimeters. This magnifier is placed directly on the film surface of the plate, which is illuminated by the viewing box.

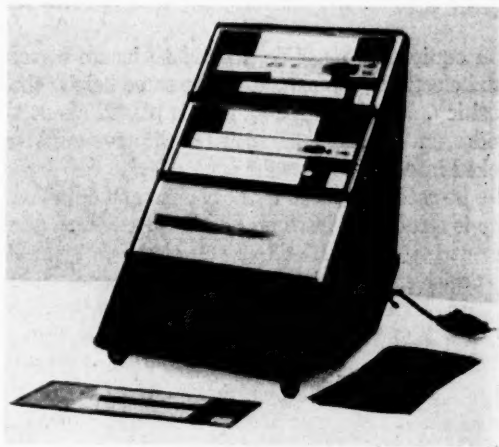


FIG. 4³³ Spectrum plate viewing box

This equipment is manufactured by the Bausch and Lomb Optical Company, Rochester, New York. On the top of the spectrograph proper, a plate is inscribed thus:

Denison University
In Memory of G. S. Jones
Class of 1895
Gift of His Sister
Miss Keturah Jones
1945

2. Procedure. The desired slit width is put in position by means of a small button which raises or lowers the slide. The position of the particular slit being used is between the heads of two screws on the face of the guides. The two-micron slit should be used when the greatest possible resolution is desired. The larger slit widths may be used when the source is weak. The time of exposure should be decreased when the wider slits are used.

³³ *Spectrographic Equipment*, p. 23.

To take an exposure, the shutter admits the light into the housing of the spectrograph. To open the shutter, the rod is pushed downward against a spring. The shutter can be locked in the open position by turning the rod clockwise while it is depressed. A counterclockwise movement will release the shutter.

The plate holder is of the book type, and 2" x 7" spectrographic plates are placed in position with the emulsion surface down. This should be done in the dark room to protect the emulsion from light. The holder fits into gibs at the top and is locked by a lever. The spectra are properly spaced by means of a scale at the right. There are seven scale divisions and ample space between each spectrum is provided by photographing a spectrum at each scale division. A diaphragm permits light to fall on the plate and to form lines only three millimeters high.

The illumination equipment must be aligned to insure correct focus. The condensing lens and electrode gap should be the same height above the optical bed as the slit position. The condensing lens is placed about three inches in front of the electrodes, and sharply focused on the slit aperture from this position. It can then be locked into position on the optical bed.

The electrodes to be used are placed in the clamps and adjusted vertically and laterally. The arc is struck by drawing a carbon rod across the gap after the direct electric current is turned on. The lower electrode is negative, while the upper electrode is positive. The time of exposures depends on the light source and slit width, but usually one, three, five, ten, or twenty seconds, as measured on a stop-watch, are sufficient to photograph the important spectral lines. When the instrument and equipment are adjusted, the slide behind the plate holder should be pulled out to expose the emulsion to the dispersed light. Before removing the plate holder, the slide should be inserted to close the back of it from light. It can then be unlocked and lifted out of the gibs. The plate is now ready for development, and the plate holder should be taken to the dark room where the plate can be removed and processed.

3. Spectrochemical Analysis. The first problem to be solved was the identification of some of the major lines in the copper spectrum. Not knowing even the approximate position of different wave lengths, it was decided to study the spectrum of an element which has only a few lines, such as mercury.

Therefore a mercury arc-lamp was used to photograph the mercury spectral lines. On the same spectrogram a copper spectrum was photographed.

Using the *Handbook of Chemistry and Physics* by Hodgman, which has wave-length tables of the insistent lines of all the elements, the mercury lines could be identified. The triplet at 3650, 3654, 3663 Å was located, and from this position the other lines were identified, as a doublet at 4046, 4077 Å, the major line at 4358 Å, and another doublet at 5769, 5790 Å was located. This procedure was the basis of further work, as the unknown copper lines could now be identified by the relative positions to the mercury lines.

Since the approximate range of wave lengths was now known, the sample copper spectrogram was studied. By using W. R. Brode's book, *Chemical Spectroscopy*, in which several copper spectra were printed adjacent to a wave-

length scale, the copper lines were identified. The reference lines of copper are in a doublet at 3247, 3274 Å. These are the main lines and always appear brightest in a copper spectrum.

Because the scale which can be imprinted on the plate was not available, the standard of reference in my work was the copper spectrum. By means of the hand magnifier, the distance between the copper lines could be measured accurately. Knowing the wave lengths of many lines and the distances between these lines, a graph was constructed on a large scale (24 x 30 inches). It showed the logarithmic relationship between wave lengths and linear dispersion. This graph is used to determine the wave length of an unknown line. The distance, d , from the unknown line to a known line of copper in an adjacent spectrum is measured. Finding the position on the curve of the reference copper line, and following along the curve from this point the distance, d , it is possible to read the

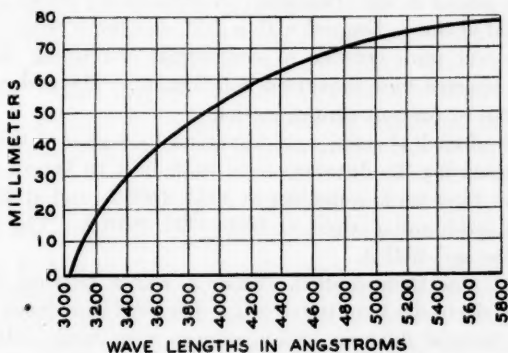


FIG. 5

corresponding wave length of the unknown line. For this instrument, a miniature graph is shown.

Before doing any analytical work, the next problem was that of making a "good" spectrogram with sharp lines three millimeters high throughout the length of the plate. The conditions which controlled the lines were thought to be these:

1. Time of exposure.
2. Slit width.
3. Voltage applied at the electrodes.
4. Type of spectrographic plate used.
5. Distances between electrodes and condensing lens, and between condensing lens and slit.
6. Focusing the condensing lens on the slit.

This problem was studied because of the poor spectrograms first made. These conditions were varied in spectrograms of copper and iron. By keeping all the conditions the same, except one, and applying the trial-and-error method, it was found that the lines were sharper and clearer when the spectrum was photographed at a slit width of two microns and at exposures of 3, 10, and 30

seconds. A current of 220 volts is sufficient for general work, since higher voltages melted the electrodes.

Two types of Eastman spectrographic plates were tried. A general plate (4-L), which is not sensitized in any region, proved to be more desirable than the III-F plate, which is specially sensitized in the region from 4500 to 7000 Å.

After several trials of experimenting with the position of the condensing lens and the corresponding focusing of the light on the slit, the most effective position was found. The focal length of the condensing lens is about three inches; therefore, if the lens is placed this distance from the electrodes and about 16 inches in front of the slit, the spectral lines photographed are longer, sharper, and more numerous.

Since good spectrograms could now be made, the method of vaporizing crystalline solids was studied. Pure carbon electrodes are used, and the crystals to be studied are placed in the lower one. A small cavity about $\frac{3}{8}$ inches deep is bored in the end of the carbon rod with a drill, and the crystals are placed in this cavity. Several pure crystals of compounds containing elements with relatively simple spectra were vaporized individually. The carbon cavity was re-bored each time to prevent contamination.

The main lines of each element were identified by means of the large graph (Fig. 5), after measuring the distances from these lines to known copper lines. The main calcium lines are in a doublet at 3933, 3968 Å, and at 4226 Å. Zinc lines are at 3303, 3345, and a triplet at 4680, 4722, 4810 Å. The main lines of barium are at 4554 and 4934 Å.

An "unknown" was then studied. Since it was a mixture of crystalline solids, the carbon electrodes have to be used. The solid was placed in the cavity and vaporized. Several different time exposures were made, adjacent to the reference spectrum of copper. Below is a print of the "unknown" spectrogram.

The wave lengths of unknown lines were determined by means of the hand magnifier and the graph. Knowing the wave lengths, it is then necessary to relate them with the elements to which they belong. The lines of calcium were easily identified—the doublet at 3933, 3968 Å, and the bright line at 4226 Å. By means of the wave-length tables the element, sodium, was found to have lines which corresponded to those in the unknown spectrum—a doublet in the red region at 5890, 5896 Å, and a bright line at 3302 Å. The lines of copper were quite faint, probably due to rapid vaporization before the spectrum was photographed. The "unknown," therefore, is composed of compounds of calcium, sodium, and copper.

B. Miscellaneous Items. 1. Timken Roller-Bearing Company. On December 26, 1945, the opportunity was given to visit the spectrographic laboratory of the Timken Roller-Bearing Company in Canton, Ohio. The chief chemist explained the use of the spectrograph in their steel business.

This company is using the diffraction grating type of spectrograph, which is manufactured by Baird Associates, Cambridge, Mass., and which has a focal

length of three meters. They have found that the spectrographic analysis is much faster and more convenient than the chemical methods of analysis. The ideal analysis requires about 6 minutes, that is, when only one or two elements in the steel sample are to be studied. A complete analysis for all elements requires 10-11 minutes.

When a certain "heat" is in the furnace, sample molds are taken from the liquid metal periodically to be analyzed, thereby keeping close check on the specifications of the "heat." When the analysis shows that all the elements are present in the correct percentage concentration, the "heat" can be poured out.

Duplicate samples are sent to the chemical laboratory and to the spectrographic laboratory. The results of these two methods agree with surprising accuracy. The former method is more accurate when the element is present in large percentages, but the latter method is accurate to 0.001% when the element is present in quantities up to about 5%.

The steel sample is received in the form of pins one-quarter inch in diameter. The one end is ground to an angle of 120° by a motor-driven drill chuck and a grinding wheel. These pins are placed in the electrode holders and are used as the electrodes. A special index keeps the pins at a constant gap of $5/32$ inches. An air blast of two inches pressure of mercury is directed onto the spark to steady it for the test.

For quantitative work the high voltage spark shows greater accuracy. The direct current is used for the arc, which is hotter and better in qualitative work with elements in low concentrations and over small ranges, but it is not as steady as the 2200 alternating current arc. The light enters the slit, passes to the diffraction grating and is dispersed into a spectrum which is focused on the photographic plate. The voltage power varies inversely as the length of exposure.

In order that the energy release is constant to give an even exposure, a pre-spark of 10 seconds allows the energy release to reach a constant point. An auxiliary target is placed above the slit which shows the reflection of the spark on the grating.

The grating type of spectrograph has an advantage in this alloy analysis over the prism spectrograph, in that any portion of the spectrum can be exposed and magnified, whereas the latter type gives a logarithmic dispersion on the plate. In steel work iron lines are very numerous and would make the analysis more difficult, but the iron triplet at 3100 Å is used as the standard reference. For quantitative analysis a densitometer is used. This studies the intensity of the lines and gives the intensity of a line in intensity ratios. The internal standard method of quantitative analysis is applied.

2. The Ohio State University Spectrographic Laboratory. During an inspection tour of the Chemistry Department of The Ohio State University on April 15, 1946, it was our privilege to have Dr. John H. Shenk, Professor of Spectroscopy, demonstrate and explain the use of the spectrographs in the laboratories at the University. The laboratory is well-equipped with spectrographs ranging from the small Littrow type to the large diffraction grating type.

On the large instruments, adjustments of the lenses and prisms is necessary

for each spectrum photographed, according to the wave lengths to be studied. For the larger instruments film is used instead of photographic plates. In analyzing the spectrograms, the film is stretched onto a holder and clamped into position in this frame. It is then placed in an instrument which magnifies and illuminates the spectra. Persistent lines of all the elements are marked on a movable scale. When the reference lines in the spectrum to be studied coincide with those on the scale, other unknown lines can be identified directly. The theory and use of spectrographs is taught in courses offered at The Ohio State University, where students work on their own analyses.

3. Denison Scientific Association. On April 23, 1946, at a meeting of the Denison Scientific Association, we were invited to discuss our work with the spectrograph. At that time, we explained the fundamentals of spectroscopy and demonstrated the small Littrow type of spectrograph. The methods involved in spectrochemical analysis and the applications of the spectrograph were discussed.

III. ACKNOWLEDGMENTS

It has, indeed, been a privilege to have been able to study the new spectrograph which was given to Denison University this year. I would like to express my appreciation to Dr. W. A. Everhart, Professor of Chemistry, for his advice and guidance in this honors project. To Dr. L. E. Smith, Professor of Physics, I want to extend my gratitude and appreciation for his assistance and direction in my laboratory work.³⁴

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³⁴ Dr. Smith has brought to my attention that David Rittenhouse, an American physicist of the 18th century, actually made a grating and good observations with it long before Fraunhofer. T. D. Cope, "David Rittenhouse, Physicist," *Journal of Franklin Institute*, V. 215, 1933, 287-297.

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SYNTHETIC RUBBERS

WILLIAM FARRAND LORANGER AND ROBERT GARFIELD ANDERSON*

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FOREWORD

We are not interested in any one particular phase of the Synthetic Rubber Program. Rather we are more concerned with the industry as a whole: its background; what led to research in the "rubber world"; the general types of rubber and a few of their properties; how research is being carried out today; and the economic and social importance of artificial rubber. These things we have endeavored to point out.

We are not willing, at this point, to take a stand as to whether synthetic rubber is better than natural rubber; so many things enter into such a question. Instead we hope to convey the idea of the magnitude of this great industry. No one can deny that synthetic rubber has done, is doing, and will do, much for the world!

SCIENCE

It split the atom, sped the plane,
Conditioned air and deadened pain,
It lengthened life and streamlined wars,
It measured molecules and the stars. . . .
Then one day lamely it announced
It didn't know why rubber bounced,
And SCIENCE, dating right from then,
Just never seemed the same again!

Richard Armour
"The New Yorker"

* The authors wish to thank Drs. W. A. Everhart and W. C. Ebaugh for their helpful suggestions in preparing this paper. The selection of material, illustrations and arrangement are all our own.

The Woodland Chemistry Prize is offered under the following conditions, and was taken jointly by William Farrand Loranger and Robert Garfield Anderson in 1946. The former's work represented chiefly what the author had done in one of the large research laboratories, and the latter's employment was in the literature of the subject. It was impossible for the jury of award to select the one contestant as superior to the other here honored.

WOODLAND PRIZES IN CHEMISTRY. Under the terms of the will of J. Ernest Woodland two prizes have been established in memory of his father, William Henry Woodland. These prizes are to be awarded annually as follows:

(a) One hundred and fifty dollars to the student, in any course in Denison University leading to the degree of Bachelor of Arts or Science, who shall prepare under the direction of the head of the Chemistry Department of Denison University, and present at the end of

I. NATURAL RUBBER: A BACKGROUND

"Although the tree *Hevea Brasiliensis*, source of most of the world's supply of crude rubber, is native to the Amazon basin in South America, it has attained its greatest importance in Southeastern Asia and the Malayan Archipelago."¹ The seeds of this tree were taken to the east by Sir Henry Wickham (in 1876) and were successfully transplanted. Today the trees are numbered in the millions, growing in Malaya, the East Indies, Ceylon, the Philippines, and in some places, in Africa.

Therefore ninety-seven per cent of the world's supply of crude rubber came from Southeastern Asia and Malaya, and all this rubber is "Hevea Rubber." The so-called "plantations" are in the form of small native holdings, and no visible attempt is made to cultivate the trees. The trees are tapped very irregularly, depending on the market value of the rubber, or the desire of the native for extra money. The estates owned and operated by European or American companies produce the best rubber.

With the expansion of the automotive industry, the rubber industry prospered. One without the other was next to impossible. Rubber was being used as insulation for wires, seat cushions, weather stripping, shock absorbers, as well as for tires. In fact over three hundred pounds of rubber go into the manufacture of a single automobile.²

It seems hard to believe that with such a tremendous supply of rubber in sight as there was in the East, that scientists would turn their eyes toward an artificial product. But that is research, always in quest of something better than what is already being supplied to the world. Probably the first important reason to start science in search of a "synthetic" rubber was the instability of the price of crude rubber.³

In 1920 the price of crude rubber was thirty-six cents, but in 1932, only three cents. Economists and scientists agreed that research in the field of an artificial rubber product leading to a stabilization of rubber prices would be a good thing.

But just exactly what was rubber was the question in their minds. The empirical formula of rubber, or more correctly of the hydrocarbon that makes

his or her junior year, the best original thesis on some phase of chemistry in its relation to industrial or every day life.

(b) Fifty dollars to the student, in full and regular standing in any course in Denison University leading to the degree of Bachelor of Arts or Science, who shall prepare, under the direction of the head of the Chemistry Department of Denison University, and present at the end of his or her junior year, the second best original thesis on some phase of chemistry in its relation to industrial or everyday life.

The purpose of these prizes is to stimulate a wholesome interest in the practical applications of the science of chemistry.

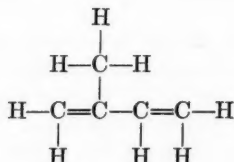
¹ "The Vanderbilt 1942 Rubber Handbook," Chapter I.

² From "Trees and Test Tubes," Wilson, p. 222.

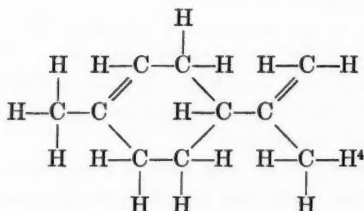
³ "The Vanderbilt Handbook." Figures from the U. S. Department of Commerce. (Figures include Latex.)

up over ninety per cent of rubber, is (C_5H_8) . To account for its high molecular weight, the formula is usually written as $(C_5H_8)_x$ in which "x" is of the order of 2000. Thermal decomposition of rubber leads to the formation of volatile products with the same empirical formula. These compounds are more highly unsaturated than rubber, either by reason of an extra bond or a ring. Isoprene and di-pentene are among these volatile products, and are represented as

ISOPRENE:



DI-PENTENE:



Since these hydrocarbons are "interconvertible," the simultaneous occurrence of both is easily understandable. Organic chemists have found that isoprene is obtained in greatest amount, and since it has the same empirical formula as rubber, and polymerizes to a rubber-like product, it is generally considered to represent the monomeric unit of *rubber*. It was Williams in 1860 who obtained a water-white liquid from rubber with the same empirical formula as rubber, which he called "isoprene."⁶

Bouchardat, in 1875,⁶ polymerized isoprene with hydrochloric acid to a rubbery mass, and for the first time the relation to rubber was recognized. Since that time the list of materials which may be polymerized to a rubber-like product has been greatly expanded. (1,3 pentadiene, 2,3 dimethylbutadiene, etc.)

It was not until 1910⁷ that Lebedeff, a Russian chemist, discovered that butadiene could be converted to a rubber-like product. Since that time the polymers of butadiene have become the most important of the synthetic rubbers from the standpoint of volume produced.

⁴ Powers, "Synthetic Resins and Rubbers."

⁵ Powers, "Synthetic Resins and Rubbers."

⁶ *Ibid.*

⁷ *Ibid.*

II. THE SEARCH FOR SYNTHETIC RUBBER

The actual chemical name for the process which "converted" butadiene to a rubber-like product, is "polymerization." Polymerization is generally regarded as the linking of a number of monomers (such as butadiene, styrene, isoprene, etc.) into long open chains. The properties of the product depend upon the extent to which the polymerization is carried out, or to the length of the chain, which is dependent upon the conditions under which the substance was polymerized.

Most of the monomers in the synthetic rubber industry are liquids, so generally the character of the material change is in becoming more viscous, even to the solid state. There are several different methods of polymerization:

1. Direct polymerization of the monomer without a solvent.
2. Polymerization of the monomer in some solvent.
3. Polymerization of the monomer in some immiscible solvent.
4. Polymerization in the gaseous phase.

Of course there is a wide variety of catalysts used in the polymerization process today. Among the most important are various oxidizing agents, aluminum, boron, heat and light are utilized.

Of the four various polymerization processes named above, number three is the most important, and the most generally used. Rubber technologists and chemical engineers know it as the "emulsion polymerization process." Previous to this, the first real advance was the use of sodium in the polymerization process. The reaction was difficult to control, however, took periods of time that were exceedingly long for profitable commercial preparation, and frequently led to variations in the quality from batch to batch.

Gradually, however, the use of sodium was developed, with the result that even today large scale production by this method is obtained in Russia. As originally practiced, metallic sodium in the form of a wire was used. It was also tried in other forms—powders, etc. The use of other controlling chemicals, which they termed "regulators," aided in bringing the process of sodium polymerization to a high state of efficiency. Examples of these so-called regulators are cyclic and unsaturated ethers, aliphatic ketones, unsaturated aldehydes, ammonia, and various amines.

To give concrete evidence of the differences between the emulsion polymerization process as used in the United States, and the sodium process used in the Russian plants, a detailed description of each method follows:

Impure butadiene (containing at least seventy-five per cent butadiene) in a batch of seventeen hundred liters is reacted in a vessel lined with a paste of clay or glycerol. The sodium, used as a catalyst, is introduced in the form of a coating on seventy-two iron combs. The charge is heated to thirty degrees centigrade and the temperature is raised five-tenths to eight-tenths of a degree centigrade per hour for eight hours. The reaction is exothermic, and cooling is necessary, the temperature is never allowed to exceed sixty-five degrees centigrade under a pressure of eight atmospheres. The reaction is complete after one hundred twenty hours.^a

^a Powers, "Synthetic Resins and Rubbers."

The very fact that it does take such an amazingly long time for the reaction to go to completion is the basis for the criticism of this method.

Since the introduction of emulsion polymerization in 1933, many improvements have contributed to its almost universal adoption. Butadiene and copolymerizing monomers are formed into an aqueous emulsion by the use of wetting agents or soaps. Emulsifying agents which may be used are sodium oleate and stearate, sulfonated fatty acids, etc. In order to stabilize the emulsion, small amounts of a protective colloid (gum arabic, gelatin, starch, dextrin, albumin, etc.) are added. To the stabilized emulsion, a polymerization catalyst is added, together with a small amount of a modifying agent. The polymerization catalysts are generally oxidizing agents such as alkali perborates and sulfates, hydrogen peroxide, etc. The oxidizing agent seems to have a directing effect on the course of the polymerization, both in favoring the formation of linear polymers and in determining the degree of polymerization. For this purpose carbon tetrachloride, hexachloroethane, sodium cyanide and other organic halide compounds are suitable.

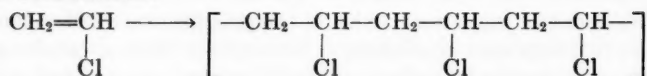
The emulsified reaction mixture is then warmed to forty to sixty degrees centigrade to shorten the time of reaction. In general, reaction time is from ten to fifteen hours, depending upon the temperature applied.⁹

Quite a difference in time between the two methods—almost one hundred hours or more! Since the length of time is liable to cause variations in the quality of the batch, and on the quality of the batch depends the physical properties of the polymer, it is without a doubt that the emulsion polymerization method of the United States far surpasses the antique sodium polymerization method of the Soviet Union.

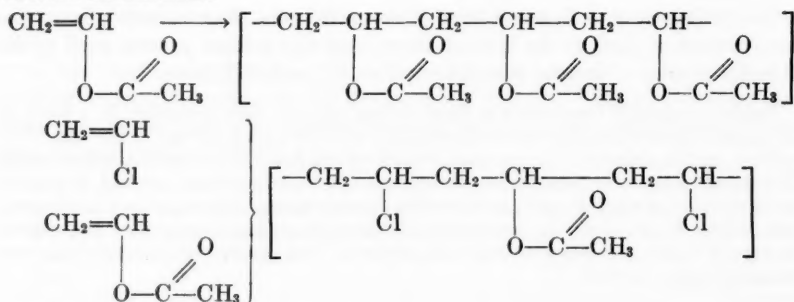
Now that scientists had mastered the art of polymerization, they began to wonder about co-polymerization. For example, what would be obtained if a monomer and a polymer were reacted together, or perhaps two polymers, or more simply, more than one monomer?

For example, the polymerization of vinyl chloride and of vinyl acetate, and in turn the co-polymerization of those polymers:

VINYL CHLORIDE:



VINYL ACETATE:



⁹ *Ibid.*

This is not to say that mixing two monomers will inevitably produce a copolymer—just certain ones produce valuable copolymer products. Very specific processes must be used which are often hindered by traces of various compounds such as copper or amines.

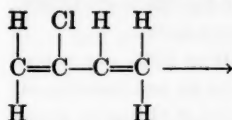
III. THE MAIN TYPES OF SYNTHETIC RUBBERS

There are several distinct phases in the effort to produce synthetic rubbers:¹⁰

1. The "academic phase." The attempt of chemists in the search for the composition of rubber, and their attempts to break it down into its monomers.
2. The discovery of coal and limestone as useful starting materials. (Neoprene.) Along with this came improvements in the process of polymerization.
3. A stimulus produced by the ridiculously high price asked for natural rubber at certain periods. Many countries entered into the quest for an artificial rubber at this point.
4. The development of many rubber manufacturing establishments in various countries.
5. The production of large quantities of artificial rubber.

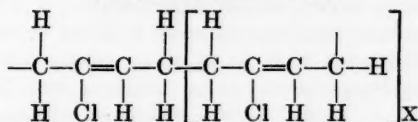
From out of all this have come several different types of synthetic rubbers. We doubt if the average layman realizes the number of varieties, and how closely they resemble one another. Few realize that as early as 1917 the Germans produced several thousand tons of methyl rubber.¹¹ From that time on, the synthetic rubber research widened its scope by leaps and bounds to what it is today.

The first commercial synthetic rubber was NEOPRENE. This made its "debut" in 1931¹² having been developed by the E. I. du Pont Company. Neoprene is a polymer of 2-chloro-1,3-butadiene. This is known to the organic chemist as "chloroprene."



CHLOROPRENE

(POLYMERIZATION)



There is a very interesting story behind the development of Neoprene. It is so characteristic of the way in which things can be interpreted as to suit one's mind or way of thinking, especially in science.

¹⁰ Harry Barron, "Modern Synthetic Rubbers."

¹¹ Powers, "Synthetic Resins and Rubbers."

¹² Powers, "Synthetic Resins and Rubbers."

Rubber's high price and enormous consumption were attracting attention of chemists, and Dr. Elmer K. Bolton, director of Chemical Research of DuPont, became convinced that butadiene was not the only derivative of acetylene that held commercial possibilities as a rubber substitute. At a chemical meeting in Rochester in December, 1925, he heard Father Julius Nieuwland, professor of Organic Chemistry at Notre Dame, read a paper on some unusual reactions of acetylene. A minor point raised by the priest-chemist fitted into the thinking of this vigilant research Director and "clicked!" Based upon some highly original discoveries Nieuwland had made—with no thought of synthetic rubber—the du Pont chemists, under the leadership of the brilliant Wallace Carothers, developed CHLOROPRENE. This closely resembles isoprene. Where isoprene carries a "CH" group, chloroprene has a chlorine atom attached.¹³

As isoprene polymerizes to rubber, so chloroprene "linked together" becomes Neoprene. It was not until 1931,¹⁴ however, that this product was offered to the public. Perhaps many people still remember the trade name under which it first appeared—"Du Prene."¹⁵ In ten years or so, Neoprene has become very widely accepted. There are few mechanical processes which do not employ it in some form or other, either as a part of a finished product or in the process of manufacture.

Neoprene, in compounding, can be handled in much the same way as can natural rubber. It can be milled or mixed on ordinary open rolls. Synthetic rubber with Neoprene as a base can be mixed in large batches, which is a definite advantage over the other types—for they require a modified technique. Even though it is mixed in a large batch, it retains a consistent quality.

To be more specific, it is necessary to give some of the physical and chemical properties of Neoprene: It has a specific gravity of 1.25. It is fairly resistant to organic solvents, but readily soluble in toluene and certain fractions of petroleum containing aromatic constituents. It easily resists diffusion of gases; is resistant to ozone, flame, oxidation and sunlight, dilute acids and alkalies, fatigue and abrasion. Neoprene's greatest uses are as wire and cable insulation, tank linings, gaskets, hose, laboratory tubing, and conveyor belts.

Neoprene had a peaceful existence up until 1941. As late as 1940, the annual production of synthetic rubber was four thousand long tons as compared to six hundred and eight thousand long tons of natural rubber used in the same year.¹⁶

But with the attack on Pearl Harbor, the picture changed. Suddenly the United States had been completely cut off from her main supply line of natural crude rubber; in fact, ninety-seven per cent of it! There were many opinions on how long America could hold out on what she had in her warehouses along with what little the synthetic rubber industry could produce. It was indeed a great problem, but by March of 1944, Rubber Director Bradley Dewey was able to write: "During February, fifty-three thousand long tons of synthetic rubbers were produced and we (the synthetic rubber industry) are now making

¹³ Haynes, "Rationed Rubber and What to Do About It."

¹⁴ Barron, "Modern Synthetic Rubbers."

¹⁵ From "The Vanderbilt 1942 Rubber Handbook."

¹⁶ "Chemical Industries", LV, (September, '44), 388-92.

synthetic rubbers at a rate greater than that at which crude rubber was consumed in this country in any year prior to 1941."¹⁷

The chief synthetic rubber being manufactured in this country is G.R.—S. (Government Rubber—Styrene.) It is also known as "Buna—S." (Derived from *butadiene* and sodium's Latin name, *natrium*: *bu-na*. This indicates the use of the sodium polymerization process previously mentioned.)

G.R.—S. is made in the form of an emulsion. When the reaction has reached the desired stage, the unreacted butadiene and styrene are removed, the copolymer is coagulated with a salt or an acid, washed and dried. In this state it is made ready for distribution to other places where it is compounded into the products that we know.

Some of its physical and chemical properties are: specific gravity of 0.94; acted upon by many solvents, such as methyl ethyl ketone, acetone, cyclohexanone, monochlorobenzene, and several other chlorinated hydrocarbons. Buna—S has great resistance to heat, aging, oxidation, abrasion, and cold. This type of artificial rubber is used as a general purpose rubber replacing natural rubber. Its properties are little better than natural rubber, and some are even worse. Very probably the existing cost at the time of need would determine whether a manufacturer would want natural rubber or G.R.—S.

Other noteworthy types of synthetic rubber are *Thiokol*,¹⁸ *Buna N*, *Hycar OR*, and *Chemigum*. The Thiokol Corporation has just recently made public that "Thiokol" is an organic polysulphide, made by a reaction between a dihalide and an alkali polysulphide. "Buna N" is manufactured by the copolymerization of butadiene and acrylonitrile in an aqueous emulsion. "Hycar OR" is merely a butadiene copolymer. "Chemigum" is a modified diolefin copolymer. All these are on an equal footing when it comes to a matter of solvents. They are all attacked by the same ones: ketones (methyl ethyl ketone, acetone, etc.) chlorinated hydrocarbons, and other solvents such as butyl and amyl acetates. "Thiokol" is especially resistant to oil and to certain very destructive solvents such as the combination of ethylene dichloride and sodium tetrasulphide. "Thiokol" is not adaptable to tire manufacture. It has good flexibility, and high resistance to ozone. It makes a perfect oil seal, as it flows under heat to fill in tool marks, etc.

"Buna N" is resistant to alcohols, heat aging and oxidation. It is particularly resistant to the action of gasoline, petroleum, and aliphatic hydrocarbons in general. It surpasses natural rubber in heat resistance, abrasive qualities (resistance), and ozone resistance. Because of this, similar to "Thiokol," its greatest use is as an oil seal, in hose for filling stations, and rubber covered rolls. However, "Buna N" is poorer in elasticity and insulating properties.

"Hycar OR" has physical and chemical properties that vary over a wide range due to variations in compounding. Generally, compounds made from "Hycar OR" are heat and abrasion resistant. They have superior resistance to mineral

¹⁷ *Ibid.*

¹⁸ "Chemical Industries", LV, (September, 1944), 388-92.

oils and fats. It is inferior to natural rubber in elasticity, tear resistance, and rebound. Like the others, it is best used in hose manufacture, as gaskets, and wherever resistance to mineral oils is important.

"Chemigum" is especially resistant to fats, oils and petroleum. It is highly resistant to oxidation, heat aging, and fatigue. It possesses the special characteristic of "self sealing," and is therefore important in the manufacture of self sealing gasoline tanks, especially since it is resistant to gasoline also. Its other uses are general ones: hose, packings, gaskets, and special oil seals.

This is not a complete list by any means. There are many more types, such as "Vistanex" and "Koroseal," but practically all of these rubbers overlap in their major properties. The list, comprehensive as it may be, gives an idea to the layman of vast numbers of synthetic rubber properties and uses in comparison to natural rubber.

There is one more type of synthetic rubber that deserves especial notice. This rubber is called "Silastic." It is now being manufactured by two companies, Dow-Corning Chemical Company, and General Electric. "Silastic" is the latest group of high polymeric organo-silicon oxide polymers. The temperature limitations of natural rubber have long been realized, so trends in research in recent months have been to perfect an artificial rubber with greater thermal resistance. Silastic rubber is just such a product—remaining elastic and compressible at temperatures of 150 degrees centigrade for long periods and can be used at temperatures up to 250 degrees centigrade for certain applications. "Silastic is not a mixture of an organic rubber with an organo-silicon product, but is the result of combining organo-silicon oxide units into high polymer molecules having elastic properties. Silastic, because of its largely inorganic origin, retains its rubberiness at temperatures which organic elastic materials are ordinarily unable to withstand." Silastic opens new doors in rubber-engineering, wherever the problem can be solved by a material which retains its compressibility at elevated temperatures. By the way, Silastic also retains good flexibility at temperatures far below zero. A tremendous advance in the rubber industry since 1931, is this development of Silastic.¹⁹

IV. RESEARCH AND SYNTHETIC RUBBER

Research in rubber has been carried on ever since Charles Goodyear's discovery, in 1839,²⁰ that raw rubber could be transformed by heating with sulfur, from a soft, yielding mass into a slightly plastic material. Scientists have worked long and hard. All kinds of materials have been tried, some successfully, others unsuccessfully, in the processing of artificial rubber. Research has been carried on extensively, always one step ahead of manufacture, so vitally important is this product called "synthetic rubber."

This work on artificial rubber has been carried on by some sixty-eight different companies. Some of the companies are petroleum refineries, others are manufacturing chemists, still more are rubber-producing concerns. The larger

¹⁹ From personal correspondence with John Church, of the Dow Corning Corporation.

²⁰ Powers, "Synthetic Resins and Rubbers."



FIG. 1. Synthetic rubber: Its progress!

companies supporting regularly staffed rubber research laboratories are such companies as the E. I. du Pont de Nemours Company, the Thiokol Corporation, the Goodyear Tire and Rubber Company. The government sponsors many

projects. Research is carried on by many other companies, who because of their trade, are naturally interested in the success of synthetic rubbers. The automotive industry is the basis of much research from this angle. The General Motors Research Laboratories, the Bell Telephone Laboratory, the Chrysler Corporation, the Dow-Corning Corporation, and the General Electric Laboratories. It is the honest, factual, scientific research that is carried on in laboratories such as these that produce results. To describe adequately such a laboratory, and to give an idea of how rubber research is carried out, we would like to include in this report, something of the work done in the General Motors Research Laboratory in connection with artificial rubber.

Rubber research is not entirely dependent on a knowledge of chemistry, but rather a firm grasp on the ideas of physics and chemistry. It was surprising to find that an industrial laboratory was not the white-tiled, sparkling glass-wared, chrome-plated set-up that had been imagined. Rather, a mixture of strange looking machines, large ovens, carbon-black stained benches, and noisy presses.

In a laboratory of this type, every make of synthetic rubber is under test. There is a constant search in the automotive industry for a synthetic rubber that will surpass all other types of artificial rubber, and even excel natural rubber in every respect.

Samples of synthetic rubber compounded from various polymers such as chloroprene, Buna—N and Buna—S are received from various companies, labeled and prepared, and then subjected to a series of tests according to A. S. T. M. Standards to determine their various physical and chemical properties. Something of the General Motors laboratory and its organization may be given. A. J. Kearfott is in charge of the Rubber-Plastics laboratory; R. K. Williams, his assistant; and four or five laboratory assistants. The laboratory itself was not very large, but the equipment was well adapted to the space allotted. There were large work benches, two large ovens, one large and two small hydraulic presses, a cold storage box, a Scott Tensile Machine, and several pieces of special equipment to run simulated service testing to give more background of the synthetic rubber as to their end use in service. Of course, there was the regular laboratory bench with its array of organic and inorganic chemicals and reagents, for once in a while we were asked to make an analysis. Most generally our work consisted in determining the physical properties of the sample. Yet the chemistry of the laboratory could not be denied, for the physical properties depend so much on the chemical constitution of the rubber.

Frequently we were asked to make various molded articles such as gaskets or thermocouples or oil seals. This was more or less routine work, yet important for the novice to learn the processes in compounding and curing of the synthetic. Perhaps the most interesting and inviting piece of equipment in the laboratory was the mill. There were two of them—one large enough to have a five pound capacity; the other, with a capacity of one hundred grams. Perhaps the best way in which to describe such a mill is to say that it looked just like the "wringer part" of a washing machine. Steam or cooling water, whichever was needed,

was passed through the rolls, and the basic polymer such as Neoprene was made to form a band around one roll, adding the various chemicals to the band thus formed. Compounding is the process of converting the basic constituent such as neoprene to a rubber-like product which can be "cured" into the substance the layman terms "rubber." Curing is the process in which heat and pressure are applied to the compounded rubber thus allowing vulcanization and conformance to the mold shape to take place.

The compounding of synthetic rubber from bases such as Neoprene, Thiokol, etc., supports a chemical industry of a size hard to imagine. There are over three hundred different types of chemicals used in compounding; some to suit the differences in the constituent, others perhaps just the type for certain temperatures or pressures, and a hundred other reasons. For example, sulfur and zinc oxide are used in tremendous amounts in the compounding process to improve the physical properties of the finished rubber article. "Captax," a commercial product known to the organic chemist as mercaptobenzothiazole, is used as an accelerator, thus cutting down the curing and processing time. Stearic acid constitutes an ingredient of almost every compounded rubber to activate the chemical and to gain its full effectiveness. A partial list only of chemicals used in the rubber industry shows more than 200 of these in use.²¹ "Gastex" is a trade name for carbon black obtained from fuel gas. It and other types of carbon black are added to produce a high tensile strength and high wear resistance so necessary in the manufacture of tires and mechanical goods. This is the reason that most of the rubber produced is black in color. Special pigments must be used in order to obtain such specialties as "white side-wall tires." Clays and other inorganic chemicals are frequently added to so called "medium and low grade" rubbers to reduce their cost and give certain desired physical properties. Regular formulas were worked out and given to us to compound, cure, and test. Upon the laboratory assistant fell the responsibility of accurately fulfilling the formula.

Neoprene Type "E".....	100 parts
Stearic Acid.....	.5 parts
Phenyl-beta-Naphthylamine.....	2 parts
Extra Light Calcined Magnesia.....	10 parts
Thermax (carbon black).....	100 parts
"Circo" Light Oil.....	5 parts
Sulfur.....	1 part
Zinc Oxide.....	10 parts

The Neoprene is the base. The stearic acid and the "Circo" light oil act as "softeners." The magnesia, and the zinc oxide act as vulcanizing²² agents. The phenyl-beta-naphthylamine acts as an anti-oxidant and prevents heat deterioration. The "Thermax" gives good tensile strength and high resilience.

After such a batch had been weighed, it was milled into a black, gummy-looking mass. This material was then cut into pieces and placed in a metal

²¹ *Chemical and Engineering News*, XXII (Nov. 1944).

²² Vulcanization: changing from plastic to an elastic state.

(usually steel) mold measuring about five inches square by one-eighth of an inch thick. This mold was then heated under pressure. (The temperature and time vary with the type of basic constituent and accelerator added; for example, most Neoprene compounds were subjected for 15 to 45 minutes at a temperature of three hundred degrees Fahrenheit.) Curing pressures are in the neighborhood of 1000 psi.

Such sheets of rubber were called "test sheets." They were carefully labelled, and tested with a Shore Durometer to determine the hardness of the rubber. This hardness number was the key to the A. S. T. M. specification to which the rubber was to be tested.

Next the test sheets were cut into various test shapes by means of special dies made according to specifications. The shapes were determined by the type of test they were to undergo. For example: a rectangular shape for oil and air aging tests; a button shape for compression set tests; a "dumbbell" shape for tensile tests; and a very narrow rectangular shape for cold tests.

It was in reality an act of scientific genius to devise the number of different tests that synthetic rubber had to undergo and survive. The physical properties of natural rubber that had to be met by artificial rubber make up a long and tedious list of which the most important are:²³

1. Great flexing endurance, withstanding considerable distortion without injury.
2. High tensile strength which reaches about thirty thousand pounds per square inch at cross section on break.
3. An elongation which can be adjusted to almost any desired length over a very wide range.
4. Great range of properties from soft to hard rubber.
5. A high capacity for absorbing energy.
6. Resistance to chemicals is outstandingly good.
7. High resistance to abrasion.
8. High dielectric strength makes rubber a splendid insulator.
9. High coefficient of friction with almost all surfaces.
10. Excellent water-proofing abilities.
11. Readily molded.
12. Available in a liquid condition as a cement.
13. A low permeability to most gases and liquids.

A tremendous challenge to men of the synthetic rubber industry! What would Neoprene or Chemigum do? Would it hold up in a tire, or would it become soft under the scorching heat of friction? Could it be molded, and would it make a good insulator? These are the questions that research answered!

The tests were conducted in a very rigid manner. Temperature condition had a plus or minus one degree tolerance for all aging. Synthetic rubber that could stand the severe tests that were required was good artificial rubber.

The fluid aging test: This was perhaps the most interesting of all the tests.

²³ Barron, "Modern Synthetic Rubber."

It determined the effect of oils, antifreezes and other fluids on the artificial rubber. For example, did the rubber contract or expand; did the rubber have more or less volume; did it crack; was it chemically inactive in the fluid? These are a few of the things in which the laboratory was interested in determining. There were printed data sheets for the work, and neatness and accuracy were impressed upon us.

The various shapes were wired together and immersed in a constant temperature bath. Some samples were left immersed for one hundred hours, others from two hundred to four hundred hours. The temperature (which was always recorded automatically) varied from room temperature up to three hundred degrees Fahrenheit.

When the required time had elapsed, various tests were conducted on the "aged" samples. The *dumbbell-shaped piece* was subjected to a tensile test on a Scott Tensile Machine. This machine is unique in that it not only reads the tensile strength of the sample directly, but records the per cent of elongation of the rubber. Such facts are vital to industries which manufacture rubber products that demand good tensile strength and a high percentage elongation. One concrete example is in the manufacture of ordinary rubber bands. Few realize the research that went into the process before synthetic rubber bands were released to the wholesale and retail markets.

The two by one inch *rectangular sample* was weighed in air and then in water, the difference being equal to the *volume*. This very important figure would tell if the rubber had shrunk or swollen. Supposing an automobile dealer needed a good gasket to use in connection with hydraulic brakes or a fuel pump line. He would want a rubber that was not affected chemically by petroleum, and one that swelled, not shrunk in it. It would be a poor gasket indeed, should it shrink and allow leakage. That is why the figures in this connection had to be accurate—which called for an ability or skill in connection with a sensitive chemical balance.

The compression set test: Clean, chrome-plated plates are used to provide a smooth surface. The rubber button, usually one half inch in height, is set on the plate between two steel shims or spacers. The shims are the desired compression height, with a plus or minus tolerance of one thousandth of an inch. Pressure was then applied, until the test sample was compressed to the desired height. The sample was heated in an oven for twenty-four to four hundred hours. When removed from the oven and the fixture and allowed to cool for exactly thirty minutes, the height was measured with a micrometer gauge. Knowing the original height and the final height, the per cent "set" could then be readily calculated. This type of information is of special importance where the rubber was to be used as a shock absorber. The capability of rubber to "regain its size and shape after deformation, especially by compressive forces, is called resilience."²⁴

The cold test: The four by one-quarter inch strip of synthetic is bent into an

²⁴ Webster, "Collegiate Dictionary," Fifth Edition, p. 848.

oval shape and placed in a cold storage box. Generally the time of the test was five hours at forty degrees Fahrenheit below zero. The test samples are kept in the oval shape by a special fixture, which can be closed tighter, pressing the rubber ends closer together. When the allotted time is up, the special fixture is pushed together, and any artificial rubber that cracks or shows definite signs of weakness is rejected. A very recent example of such knowledge in application is the Navy's "Operations Frostbite." The seals used in the searchlights are of artificial rubber—artificial rubber which must not crack under the arctic temperature conditions. This is why such tests are made.

This is not synthetic rubber research in its entirety. It is only the one small phase of commercial testing that we have personally come in contact with at the General Motors Laboratory. Some companies are very interested in the chemical side of the research—the developing of new chemicals, or better combinations of chemicals, to use as activators, accelerators, antioxidants, etc. The United States and Mexican Government have spent much time and money on the "guayule" plant which yields an artificial rubber very similar to other natural crude rubber. This "guayule rubber" is still inferior in many ways to natural rubber and to synthetic rubbers such as Neoprene or Thiokol.

There is still room for reasonable scientific disagreement over many of the processes for making rubber that are being developed. It is quite possible, even likely, that before much of the synthetic rubber now planned is produced, better processes will have proven themselves. In any new industry the processes of today are outmoded by the processes of tomorrow and tomorrow's by those of the next day.²⁵

V. THE IMPORTANCE OF SYNTHETIC RUBBER

In his report on synthetic rubber as *America's Achievement in Chemical Engineering*, Sidney D. Kirkpatrick said:²⁶

On the shores of the Bay of Bengal, at least up until a few years ago, the natives still cultivated the straggling indigo plant for its yield of natural dyestuff. Chilean nitrate helped fight the war. Vanilla and tonka beans are imported from Latin America and, at last reports, the silk-worms of the Orient were still spinning their cocoons while natural camphor was being tapped from nearby trees. All this is happening despite the fact that equivalent products of chemical synthesis have long been available of more uniform quality and usually at lower price.

Is there any parallel here in the case of natural versus synthetic rubber? What is happening and what is going to happen to the huge rubber plantations of the world now that America, Germany and Russia have built great chemical industries that can turn out a synthetic product, equivalent at least in volume, perhaps in quality and possibly in price? This is indeed a difficult question. It involves social and political as well as economic considerations.

There was a time when it seemed certain that the Japanese would never relinquish control of the Far Eastern plantations without completely destroying them. They knew that our civilization as well as our war machine rolled on rubber and they could not take the chance that such a military weapon might fall into our hands. Now the situation has

²⁵ From Baruch Report, September 10, 1942.

²⁶ Kirkpatrick, S. D., *Looking to the Future of Synthetic Rubber, Chemical and Metallurgical Engineering*, December, 1943.

changed. Our synthetic rubber industry has made us no longer dependent on the Far East so our enemies would have gained little or nothing by wantonly destroying the plantations.

If we may assume that a large part of the Far Eastern plantations will survive and not revert to jungle, and that their war-damaged plants can be quickly reconstructed, then we can expect a post war capacity of over one million tons of natural rubber. Add to this another million tons of the synthetic rubber and we have more than twice as much rubber as the world consumed, even in the peak year of 1941. Thus there must be a tremendous increase in the use of rubber products or we must face the prospect of either the natural or the synthetic rubber industry—or both—operating at less than fifty per cent capacity.

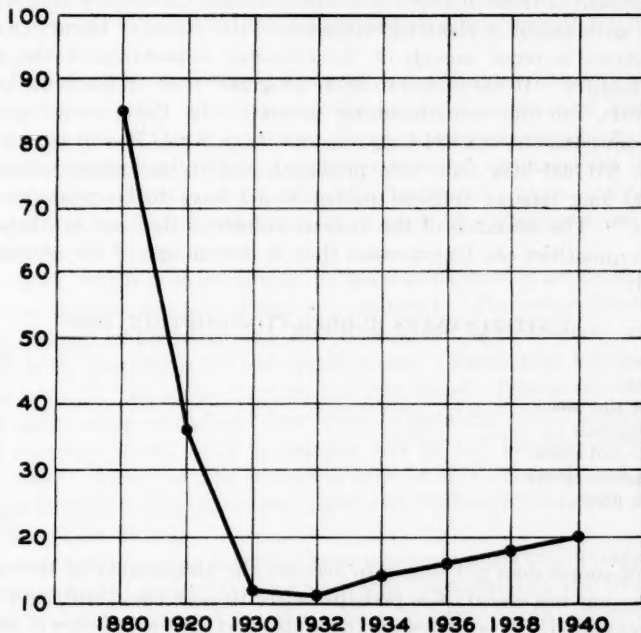


FIG. 2. The Instability of the Price of Rubber

We do not intend to include in this report a discussion on which system would be better socially, politically, and economically. Rather, we will try to point out what the synthetic rubber industry has done in the past few years, economically and socially, for the people of the United States. The question that has been troubling many, whether we should have artificial or natural rubber, is one that will work itself out in time through proper legislation.

First of all, it has provided a tremendous industry for the Northeast, Southwest, and Southern Pacific areas of our country. Such industries as the production of butadiene and styrene, two of the most valuable constituents of synthetic rubber, are among those located in those areas. The processing of rubber (which includes first the compounding) uses such a tremendous amount of such a long list of chemicals, that a virtual chemical manufacturing industry is maintained

to supply the artificial rubber industry. Then there are the actual plants where it is manufactured; there are salesmen, wholesalers, and retailers. It is a tremendously large, and all-American industry! It was developed by and for America. It creates many jobs that were not available five years ago, which means much to our economic system at present.

Perhaps the greatest single investor in the synthetic rubber program is the United States Government. Investments made up to March 1945 were totalled to the amount of three-quarters of a billion dollars. This very fact alone, plus that the government is planning expansion to the extent of twenty-two million dollars more, is proof enough of the economic importance of the synthetic rubber industry. It has ceased to be a "program" now—it has branched into a nation-wide, full-time manufacturing industry. In 1944, as a direct result of these investments, 668,834 long tons of "Buna S" (G.R.—S) were produced. In 1945, 881,000 long tons were produced, and technologists estimated that 1,054,000 long tons of artificial rubber would have to be produced to meet demands.²⁷ The demands of the various industries that use synthetic rubber today in quantities can be expressed thus in percentages of the amount (total) supplied:

UNITED STATES RUBBER CONSUMPTION, 1939²⁸

Tires.....	63.6 %
Tubes.....	9.2
Other tire uses.....	3.1
Mechanicals.....	9.0
Boots and shoes.....	6.8
Druggists, sports.....	5.0
Other uses.....	3.3
	100.0

This of course does not take into account the abnormality of the war years. Since the war has ended, it is perhaps better to note the distribution during a peacetime period. It is interesting to see also that the tire industry at that period used approximately seventy-four per cent of all the rubber produced.

To give a little sidelight on the peacetime angle, it is extremely interesting to see how rubber was used in producing the machines of war:²⁹

Heavy bomber—1,825 pounds.
 Medium bomber—804 pounds.
 Pursuit plane—255 pounds.
 Scout car—306 pounds.
 Light tank—489 pounds. (Tracks alone use 317 pounds.)
 Gas Mask—1.11 pounds.
 Gun Carriage—66.1 pounds. (37mm gun.)
 Gun Carriage—82.1 pounds. (57mm gun.)
 Two and one-half ton Army Truck—446 pounds.

²⁷ *Chemical and Engineering News*, XXIII, (April 10, 1945), 627.

²⁸ Barron, "Modern Synthetic Rubbers."

²⁹ Wilson, "Trees and Test Tubes," 222.

Motorcycle—10.7 pounds.

Pair of ski boots—0.31 pounds.

Pair of hip-boots—2.62 pounds.

Ten ton Pontoon Bridge—3,644 pounds.

Aircraft tires range from 17 to 100 pounds apiece.

165,000 pounds of rubber go into making a 40,000-ton battleship.

Then when we read in the newspapers that there had been a thousand-bomber raid over Berlin, try and visualize the amount of rubber! If it had not been for the synthetic rubber industry, and its tremendous advance in twenty-four months, what would have ordinarily taken perhaps a dozen years to do, it might not have been so easy to do!

If this is not enough concrete evidence in behalf of the importance of artificial rubber, then pick up any good scientific journal or magazine, and invariably you will find one or more articles on such things as recent developments, or historical facts, or production difficulties, or any number of various points of interest to men of science. For example, there have been many articles lately on how the shortages of chemicals valuable to the synthetic rubber program could cripple that industry.

Carbon black, which is so important in the manufacture of artificial rubber, nearly caused an acute shortage of that vital material. The carbon black shortage was so critical in February and March of 1945³⁰ as to interfere with tire production both quantitatively and qualitatively. Production has increased forty-three per cent³¹ since the shortage has been eased. The government has sponsored construction of carbon black units. Operating at full capacity, the estimated monthly carbon black production will be one hundred twenty-two million pounds.³² This would be enough to care for all U. S. needs and provide about seventeen million pounds a month for export,³³ primarily to Russia, France, Belgium, and Great Britain.

Though most attention has naturally been centered on the major components (such as carbon black) studies have revealed that approximately two thousand³⁴ component materials are used by the rubber industry (both synthetic and natural) any of which at any time or another, could present problems of supply and demand.

Legislative trends as late as March, 1946, show that ample rubber for war or peace is the goal of the United States Rubber program. Definite action has been taken along the lines to transfer much of the government controlled holdings to private ownership and operation. Various committees have recommended that the synthetic rubber industries maintain a production of at least a third of the total amount of rubber needed. According to a recent article in the *Ohio State Journal* (March 18, 1946) John W. Snyder, director of the Office

³⁰ *Chemical and Engineering News*, XXIII, (Aug. 10, 1945).

³¹ *Ibid.*

³² *Ibid.*

³³ *Ibid.*

³⁴ *Ibid.*

of War Mobilization and Reconversion, has recommended that although the synthetic rubber industry be privately controlled, the government be responsible for "broad research programs for quality and cost improvements in the synthetic rubber field." He also recommends that private industry do so, but without any admonitions whatsoever. The private industries of America will always carry out research for "quality and cost improvements," regardless of whether it is in the synthetic rubber field or any other field. It is impossible to separate good business and successful enterprise from scientific research!

Although every billboard and newsreel, newspaper and magazine, have proclaimed the efforts and results of the synthetic rubber industry, it is doubtful if people realize what a tremendous debt of gratitude is owed to American Rubber Technologists for their seemingly impossible struggle to produce a satisfactory artificial rubber. The synthetic rubber industry stands alongside the atomic and radio-proximity bombs as being the major contributions of American scientists to the speedy termination of World War II. Realize, however, that the atomic and radio-proximity bombs were destructive contributions, while the synthetic rubber industry was the constructive contribution of our science—Chemistry!

GLOSSARY

Accelerator.—A chemical used in compounding to cut down the processing and curing time; e.g. mercaptobenzole.

A. S. T. M.—American Society for Testing Materials.

Batch.—The quantity produced at one time.

Captax.—Trade name for a commercial accelerator, mercaptobenzothiazole.

Catalyst.—A substance that alters the speed of a reaction without undergoing a permanent change in composition. (Berzelius is responsible for the term—about 1835.)

Compounding.—Mixing various chemicals with the base component on the mill.

Curing.—Applying heat and pressure to the raw rubber to form the finished product.

Elastic.—Capable of recovering size and shape after deformation.

Emulsion.—A suspension of fine particles or globules of a liquid in a liquid.

Exothermic.—A reaction in which heat is evolved.

Fatigue.—The action which takes place in a material, especially in rubber, causing deterioration and failure after a repetition of stress.

Gasket.—Generally a plaited hemp, or fabric and rubber blended, or rubber alone, used as packing for pistons, etc.

Gastex.—Trade name for a particular carbon black compound used in the production of synthetic rubber so as to give the product tear-resistance and resilience.

Guayule.—A plant secreting a sap similar to natural rubber; cultivated by the American Government in Texas.

Immiscible.—A substance not capable of being mixed or mingled.

Long ton.—A weight of 2240 Avoirdupois pounds.

Permeability.—A term applied to a substance to denote the extent to which it will allow the passage of fluids.

Polymerization.—A process used to change by the union of two or more molecules into another compound having the same elements in the same proportions but a different molecular weight and different physical properties.

Solvent.—A material, usually a fluid, capable of dissolving or putting some other substance in solution.

Tensile strength.—The ability of a substance, as rubber, to resist longitudinal stress.

Tolerance.—The difference between two limiting sizes as a means of specifying the degree of accuracy.

Viscous.—Property of a liquid that denotes a glutinous consistency.

Vulcanization.—A process of changing rubber from an elastic to a plastic state.

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²⁵ (Many of the facts, especially about the properties of the various rubbers, are taken from our own notes on our work in synthetic rubber.)

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THE ROLE OF SUGGESTIBILITY IN SUSCEPTIBILITY TO THE SIZE-WEIGHT ILLUSION AND THE PHENOMENON OF AUTOKINETIC STREAMING

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PROBLEM

Recently the psychological problems involved in suggestibility have received increasing attention probably due in large part to the researches of Hull and his co-workers. This experiment was an attempt to study the relationship between the susceptibility to illusion and suggestibility.

HISTORY AND INTRODUCTION

As long ago as human beings can trace, suggestion has been used in some form or other. There is evidence that the Egyptians of 3000 B.C. employed hypnosis in various religious rites (6). The inhabitants of ancient Greece and Rome also used this form of suggestion. Another theory involving suggestion was that of Paracelsus (1493-1541) concerning the influence of the stars upon the health of human beings. This again illustrates the phenomenon of suggestion in that individuals believed their health to be affected by the stars and reacted according to the suggestion. Von Helmont advanced the theory of animal magnetism which was similar in nature to the belief that stars influenced one's health. Later it was found that physical cures could actually be brought about by this technique. From these historical facts, suggestion can be seen to have become even more powerful in its effect upon people. Perhaps the most outstanding name in this field is that of Mesmer from which the term Mesmerism was derived. In 1784, however, a commission was appointed, including Benjamin Franklin, to study the effect allegedly brought about by this so-called magnetism. The commission declared rather contemptuously that the effect produced upon an individual was not due to animal magnetism *per se* but by the influence of the magnetist himself. This finding set back hypnosis many years in its progress.

The beginning of the modern theory of suggestion, however, was developed early in the 19th century. Noizet (6) upon investigation formed the concept that an idea may be expressed in action as it comes into an individual's consciousness. Both Noizet and Bertrand were early workers in the development of theories of suggestion, but, according to Janet (*Psychological Healing*, Vol. I, MacMillan Co. 1925), Bertrand should be given more credit for the theory of hypnosis as a state of hyper-suggestibility. From about 1850 to 1885, not much

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work was done on the question of hypnosis and suggestibility. During the latter part of this period, however, some interest in suggestibility was aroused by the Salpetriere and the Nancy schools; but soon again science abandoned investigation in this field. Hull (14) and his followers working in the 1930's revived new interest in this problem and more and more today it is becoming one of the important experimental problems in psychology.

Experiments on suggestion can be classified generally into three major groups:

- 1.—*Ideomotor suggestion*: This may include experiments of the sensory-motor type or the verbal-motor type. The former takes advantage of the fact that a response once made tends to be repeated when a similar situation presents itself. The size-weight illusion, which is used in the present experiment, would come under the sensory-motor type in as much as the larger object tends to be judged lighter because of expectation on the part of the individual. The verbal-motor type involves the influence of previous "attitudes" or "sets" evoked in the individual by external suggestion. Hull (14) in his experiments on postural reactions has shown that expectation plays an important role in suggestibility. Here verbal suggestions tend to touch off an already present set in which case the acts resulting might be considered symbolic of the characteristic behavior of the individual in this area.
- 2.—*Prestige suggestion*: This involves the personal influence of one person upon another individual. There may be, however, cases of negative suggestibility to prestige. Hull (14), in his postural experiments, found this characteristic to be evident to a large extent when catatonic cases were given suggestion tests. Experiments concerning increased suggestibility through the influence of majority and expert opinion have also been conducted (2).
- 3.—*Suggestibility as a trait*: Numerous experiments have been conducted in an attempt to measure suggestibility as a "trait" and to study its various characteristics. It has been found that, for the most part, suggestibility decreases with age, the span between 7 and 9 years being the most suggestible stage. Reymert and Kohn (20) in using an original heat apparatus for measuring suggestibility found that in children of 5 to 7 years, 90% were suggestible, but of those between 17 and 20 years, only 17% were suggestible. Others who substantiate the view that younger children are most suggestible are Small, Gilbert, Ross, Yung, Giroud, Papov, Young, Messerschmidt, Sherman, Binet and others. On the whole, women and girls have been found to be more suggestible than men and boys as shown by Hull (6), Brown, Heron, McGeoch, Seashore, but this is probably a cultural rather than a biological difference. Extroverted women have been found to be more suggestible than introverted women. Possibly extroversion in women may be as much the result of suggestibility as of "masculine protest" in as much as a free, easy aggressiveness is culturally limited to the male in our society. The amount of extroversion and introversion in men seems to have no effect upon their susceptibility to suggestion (19).

The relationship of intelligence to suggestibility has not been established. Both positive and negative correlations have been found. A negative correlation has been found between scores in the Otis suggestibility test and the Stanford-Binet. However, no relationship was found between the degree of suggestibility

as measured by the Reymert and Kohn heat apparatus and the Stanford-Binet test (6). Porter also found that suggestibility seems to be inversely related to both scholarship and intelligence (19). Present evidence would, therefore, seem to indicate a negative relationship between intelligence and suggestibility.

Certain personality factors and their relationship to a suggestibility "trait" have been investigated to a limited extent. Baumgartner has presented evidence to indicate that suggestion may be a desirable "trait" in as much as it may be linked with other traits such as sympathy, sweet temper, tactfulness, etc. (4). She found, using a group of student nurses as subjects, a positive relationship to exist between scores on suggestibility tests and rankings in these traits by supervisors. Work of this sort has been limited in extent and is not at all conclusive. In such studies, there is always a question of whether or not suggestibility is a reflection of a generalized behavior pattern or only a specific response to the experimental situation. Brown concluded that individuals cannot be categorized into suggestible and non-suggestible groups (25). Rather he found individual variability to be more typical than individual consistency. He thinks that while some persons may be a little more suggestible than others, the correlation is so small that the probability is slight that a trait of suggestibility exists. There are too many individual differences and negative instances for any conclusions to be drawn. Prideau (1929) states that "a) suggestibility varies in different persons irrespective of the nature of suggestion and of the suggestor; b) suggestibility varies in the same person at different times and under different conditions." (25, p. 320) Binet, Pearce, Otis, Aveling and Hargreaves feel that there is a general trait of suggestibility in certain individuals (6). On the other hand, Scott, Chojecki, Okabe, Brown, Bessen, McGeoch, Stevick and Hull (6) find a zero or very low correlations between scores on the various tests which purport to measure this "trait" in people. Murphy and Murphy, Hull, Dorcus and Shaffer and Bird (6) are not at all convinced of a general trait of suggestibility among people. Since such correlations as have been found are low, it is probably inexact to speak of a person as being suggestible unless the conditions under which the experiment has been conducted are explained. There may be a number of psychological processes involved in this phenomenon. It seems, in any event, that the conditions under which the experiment has been performed are about as important as are individual and group differences.

There have been many interpretations given to and numerous studies made upon the characteristics of suggestion. As Coffin puts it (6, p. 110), "suggestion has been viewed as an instance of what Hartmann calls 'framework responses', responses taking place within and determined by a framework of internal determining factors and external stimulus conditions." He indicates two pertinent relationships. The first is that previously existing attitudes are important. Suggestion may or may not be accepted depending upon the previous experience of the individual and his "frames of reference." That is, the suggestion may not be accepted if it conflicts with the person's frame of reference. At first thought, one might think that this applied only to prestige suggestion

in which the opinion of others influences the judgment of the subject, but actually this may also apply to a subject who is responding to any test of suggestion. For example, in the "falling" experiment of Hull or any similar test of suggestibility, if the person involved is particularly negativistic, there will probably be a tendency for him to resist falling. In other words, his habit patterns of life may and probably do enter into his susceptibility to suggestion.

Second, it has been found that suggestion influences the responses made to unstructured stimuli (such as the Rorschach test) more than it does responses made to structured stimuli. Acceptance of suggestion, therefore, depends also upon the structuration of the situation (6).

The question of hypersuggestibility and of practice effects in suggestion has been investigated by Hull, Patten and Switzer. Their experimentation indicates that what appears to be hypersuggestibility, brought about as a result of suggestions given during the waking state, cannot definitely be distinguished from mere practice effects. Further experimentation elicited no support whatever for the hypothesis that response to waking suggestion brings about a state of generalized hypersuggestibility (16). Hull found evidence of a negative practice effect in the use of the Binet Weights test (14). Since his subjects became increasingly instead of decreasingly accurate on the test, practice would seem to negate suggestibility. Repetition, therefore, seems to weaken rather than to strengthen the tendency of a person to be suggestible. In the "falling" experiment, the tendency to respond to suggestion at the end of a ten-day period was one-third of its original strength. Experimentation seems to indicate, then, that indirect suggestion differs from direct suggestion and hypnosis in that there is no positive practice effect in the former while such effects are found in the latter.

Not much experimentation has been done recently with the phenomenon of autokinetic streaming. Voth has done work on it in an attempt to investigate "the individual's differentiation as to psychological reality." Voth¹ suggests that the problem of extroversion-introversion among individuals may be related in some way to the results obtained in the use of this illusion as extroverts are usually thought of as being more objective and realistic than the introvert who projects himself more into the situation and is more creative in an imaginative sort of way (27, p. 320). Various other personality characteristics have also been held to have been linked with particular types of response to this illusion. However, these relationships are largely speculative in nature, and the area is wide-open for experimental investigation. It is interesting to note in passing that at least one practical application of this test has already been made by Graybiel and Clark. During the war years, 500 fliers were tested for autokinetic streaming and all 500 of them experienced the phenomenon (7). It was thought that many flight accidents might have been caused by the fact that lights on the ground, viewed by the fliers and used by them as guides in landing their planes, may have been seen to possess apparent movement and distances might, therefore, have been misjudged thus causing accidents.

¹ Using clinical material, Sexton (23) has recently verified Voth's findings.

Hull has been largely responsible for the development of the postural-sway test used in this experiment. He has performed the experiment with numerous variations. Not only has he used body fall, but arm and head movement as measures of suggestibility. In one instance, the subject was told to stand and a hook was put on his collar unknowingly which was attached by means of a string to a recorder (kymograph). An assistant then entered the room and assumed various bodily positions. The movements of the subject were recorded, and it was found that the assistant's attitudes were imitated unconsciously by the subject.

APPARATUS

The size-weight² illusion as employed in this experiment consisted of the largest and the smallest of the Kline canisters. (The larger was 4" deep with a diameter of 3.25"; the smaller was 2.25" deep and 2.50" in diameter). Both were of the same weight thus creating the illusion that the larger bucket was heavier.

The traditional conceptions concerning the basis of this illusion are that 1.—a larger object is expected to be heavier than a smaller object and therefore a person "sets" himself to lift a heavier object, and that, 2.—the apparent difference in weights is due to the fact that the motor effort is different for the two objects. There have been various attempts made to explain the phenomenon of an individual's judgment that large objects are heavier. Flournoy thought that individuals inherit a characteristic cerebral impulse to judge weights in relation to volume (12). More popular, however, is the belief that this "set" or expectation is due to the fact that through experience we build up an association between the sizes of objects and the weights of objects. Individuals, then, become used to thinking that larger objects weigh more and therefore their muscular "set" for lifting differs in proportion to the size of the object. Some call this attitude "suggestion" but others, e.g. Myers hold that a more exact term be used, such as "unconscious inference", which is working "irresistibly" (12). This theory assumes that there is a psychic process similar to Helmholtz's theory concerning color discrimination in which an unconscious analogy is made and, through repetition, becomes automatic. When two objects are shown, then, the larger one, by contrast, is inferred to be heavier.

As to the weight impression, Flournoy thought that the S-W illusion disproved the theory of "sensations of innervation" because he believed that the energy used in lifting both weights is equal. Charpentier believed that sensations of this sort were present but were rather over-ruled by the fact that a larger area of the skin is stimulated when the larger object is lifted (12). In other words, he considered the amount of pressure on the hand to be of greater importance from the lifter's point of view than the total weight. Flournoy, however, brought about the illusion by using strings to lift the objects, so Charpentier's theory was thereby disproved (12). Müller and Schumann in their theory came to the conclusion that, as the motor set had been adjusted for a heavier object when the large one was about to be lifted, the illusory weight seems to "fly up" when lifted, and the smaller object tends to be pulled downward. Loomis found by kymographic records of the movements made in lifting the weights that less energy is put forth when the smaller weight was lifted (12), and Payne and Davis found that a smaller action potential is evident when the supposedly "lighter" object is lifted, thereby demonstrating that equal energies were not involved as Flournoy has supposed (12).

Usnadze has shown other effects. One is that of objects which are rested upon the hands (12). As in the previously mentioned experiment, the larger was judged heavier

² Hereafter, "S-W" will be used to indicate this term.

although the weights were the same. He also demonstrated a "volume illusion" by reversing the procedure used in the weight illusion. If, then, two objects of equal size but with different weights are held in the hands of a blind-folded person, the lighter one appears to be the larger of the two objects in the majority of cases.

When two objects of equal weight but different sizes are presented, there is a real difference and an illusory difference both of which are perceived by the subject and the illusion is obtained. If, however, only one object is shown, size does not seem to influence guessed weight because, apparently, there is nothing with which to compare it.

Thouless (12, p. 47-8) has worked out a theory concerning the S-W illusion stating that the "apparent weight" is a compromise between the real weight (indicated by the muscular and tactual stimuli) and the density (the cues to which must be derived both from the combined action of the cues to weight and of visual indications of size).

The nature of the S-W illusion seems to be sensorial rather than intellectual for even if the knowledge of the phenomenon is revealed to the subject, the illusion still appears, even if to a lesser degree than before. Practice seems to have little effect upon the S-W illusion and it is also interesting to note that it is hardly evident in children under three years of age. By six years, however, it is well developed and it seems to increase up to nine years at which time it begins to decline slowly.

In this experiment, the subject was given as much time as she needed to equalize the weights and the following instructions were given:

Here are two buckets and here is shot. Add shot to the lighter bucket until the two seem to you to weigh the same. This is a test in kinesthetic sensitivity and I want to see how close you can come to making the two weigh the same.

The second test used was the phenomenon of autokinetic streaming. The subject, for this part of the experiment, was placed in a darkened room alone seated in front of a table at the rear of the room. At the front of the room was placed a light-tight container in whose side a one-mm hole had been pierced. Within the container was an electric light bulb from which light passed through the small opening just mentioned. The subject was given the following instructions:

Keep your eyes fixated on the light in the front of the room. If you see any movement of the light, draw a continuous line graph of this movement as accurately as possible indicating both the direction of the light's movement and its extent. Start with your pencil at the center of the paper. If the light seems to stop, indicate this stop by an "X". If the movement carries your pencil off of the paper, start again as close to the center of the paper as you can.

The test period lasted ten minutes and indices were obtained for each subject by taking various measurements of the lines drawn and placing them in the formula developed by Voth (27). The total length of line drawn was measured (1), the maximum distance attained from the starting point (DC), the distance between the two points on the graph which were farthest apart (ME) and the number of stops, or "X's" made (S). These measurements in millimeters were put into the formula $\sqrt{\frac{L \times DC \times ME}{S + 1}}$ and from this, the index of the subject's susceptibility to the autokinetic illusion was found.

Our modification of Chevreul's pendulum, the next test used, consisted of a

block of wood on which there was a horse-shoe shaped bar. In the closed end of the bar was a section of 6-inch steel pipe 4 inches high. On the board holding the metal bar and surrounded by the pipe, two concentric circles had been pointed around a center dot. In this way, the amount of swing of the pendulum could be measured to a certain extent, and a score of 1, 2, 3 or 4 could be obtained depending upon the extent of the swing. The subject was given a string about a foot long at the end of which was a small iron bob. The following directions were then given:

This is a test to see how responsive you are to the lines of force which are present within this magnetic field. Some people are quite sensitive to them while others may not respond so successfully. Take this string and hold it so that the bob is in the center. If you are sensitive to the magnetic lines of force, the bob will begin to swing from side to side.

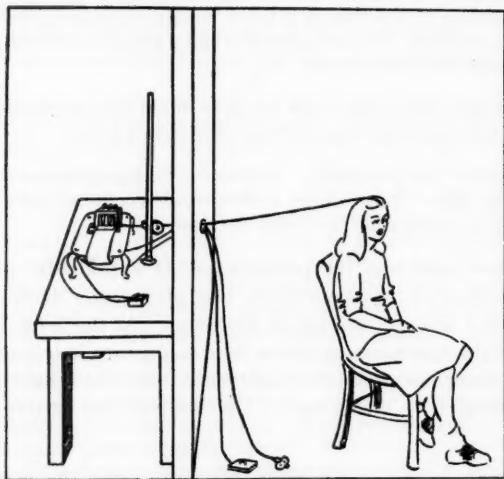


FIG. 1

Five minutes was allowed for each subject and so both time and extent of swing scores were recorded. An index was later obtained by dividing the extent by the time.

The postural suggestion test was the next used and for this the subject was placed in a chair and was blindfolded. A string was run by means of reducing pulleys from a polygraph (placed in another room to eliminate the noise factor) over the subject's head to her forehead and attached there to the blind-fold. Figure 1 illustrates the position taken by the subject in relation to the polygraph. She was allowed to sit for about a half of a minute with the polygraph running in order to obtain a record of any natural movement made by her head while in a comparatively relaxed and stationary condition. Then a phonograph record was played with the suggestions given to the effect that the subject's head was falling farther on her chest and the subsequent movement reduced by

3 to 1 was recorded by the pen on the Renshaw polygraph. Time in seconds was also marked as were the points at which the record started and stopped. The complete record lasted approximately three and a quarter minutes and contained the following suggestions:

Now I want you to relax and sit as quietly as you can and listen carefully to what I say. I want you to imagine that your head is falling forward on your chest. Please do not try to prevent any response you feel like giving. If your head feels as if it wants to fall forward—just let it go. Just sit quietly, relaxed as much as you can, and imagine that your head is falling forward on your chest. Head falling forward. Head falling forward. Head falling farther and farther forward. Your head is falling forward on your chest. Just imagine that your head is falling forward on your chest. Your head is falling forward, farther and farther forward. Falling farther and farther yet, more, more yet. Your head is falling farther and farther forward on your chest, head falling farther and farther forward, farther yet, more and more, more yet. Farther and farther forward on your chest. Very easy for your head to fall forward on your chest, very easy to do,—Just relax and listen to my voice, let your head fall farther and farther forward on your chest, etc., etc.

As two scores were obtained, time and amount of head fall, again an index had to be computed. This was done by dividing the amount of head fall by the time consumed.

The fifth part of the experiment consisted of the suggestibility section of the Adams-Lepley Personal Audit. This is a paper and pencil test for which the subjects were given unlimited time (1). This part of the Adams-Lepley test was designed to see how great a tendency there is on the part of a subject to agree with authority. The directions read:

Fifteen experts, each of whom had won success in a different field, were asked to give an opinion of the statements below. At least eight or more of the experts marked each statement below as true. Read each statement, and if you agree with the experts that the statement is true, draw a circle around "A". If you agree, but only with reservations, draw a circle around "R". If you disagree with the experts, draw a circle around "D".

Some of the questions included in the test were: "Persons of widely differing races or faiths should not marry." "No further immigration to the United States should be permitted." "Any person who really wants to work can always find a job." "Most parents know how to raise their children properly." etc.

Thirty subjects were used, all freshmen women, none of whom had taken any course work in psychology.

RESULTS

Tables 1, 2 and 3 give the results of this experiment. Table 1 includes the data obtained from the S-W illusion (the amount in grams by which each subject overestimated the weight of the smaller container); Chevreul's pendulum (the time used by the subject, the amount of swing of the pendulum and the index obtained by dividing the amount of swing by the time); and the index for the autokinetic streaming test. Table 2 includes the data secured from the head-falling experiment (time used by the subject to respond, the number of centimeters the head fell and an index obtained by dividing the distance by the time);

and the Adams-Lepley Personal Audit test of suggestibility (both the raw scores and the percentile scores). Table 3 gives the coefficients of correlation computed from the data secured in the various test parts. The correlation between the the pendulum and the test of autokinetic streaming is the only significant one, $r = 0.56$.

TABLE 1
Data Obtained from S-W, Pendulum and Autokinetic Tests

NO.	SUBJECT	S-W	TIME	AMT. SW.	$\frac{S}{T}$	AUTOK.
1	A. K.	37	2	4	2	60
2	L. S.	54	4	4	1	142
3	R. S.	80.5	4	4	1	80
4	J. C.	118.0	5	1	.2	19
5	C. O.	155.5	1.25	4	3.2	760
6	B. H.	41.5	5	3	.6	113
7	P. H.	169.5	5	1	.2	138
8	B. C.	122	5	1	.2	115
9	L. L.	56.5	5	2	.4	185
10	S. H.	143.5	5	1	.2	226
11	M. B.	158	5	1	.2	133
12	L. P.	256	5	1	.2	73
13	M. W.	148	5	2	.4	314
14	J. S.	91	5	3	.6	93
15	P. K.	156.5	2.5	4	1.6	149
16	S. B.	151.5	5	1	.2	216
17	N. K.	120	5	3	.6	78
18	H. H.	129	5	1	.2	114
19	S. A.	98.5	1.75	4	2.3	137
20	N. S.	128.5	5	1	.2	129
21	Y. R.	188	5	1	.2	267
22	J. H.	306.5	5	1	.2	132
23	M. S.	81	5	1	.2	204
24	B. D.	214	5	2	.4	395
25	P. R.	121	5	1	.2	66
26	B. S.	25	5	1	.2	12
27	E. F.	106	5	2	.4	192
28	M. N.	121	5	1	.2	4
29	A. D.	119	1.5	4	2.7	430
30	P. W.	157.5	2	1	.5	145

There was in the S-W illusion quite a range in the over-estimations made by the subjects. The lowest was 25 gms. and the highest, 306.5 gms. The autokinetic streaming part of the experiment also shows wide variance, the least movement giving an index of 4 and the greatest index being 760. In the head falling experiment, quite a few reacted to a maximum degree, the head falling as far as the play in string would allow. It is interesting to note that of the subjects whose heads fell completely forward (or as far as possible), only one required the maximum time (3.25 minutes). As is seen in Table 2, nine of the thirty subjects

TABLE 2
Data Obtained from Head-Falling and Personal Audit Tests

NO.	SUBJECT	TIME	AMT. (CM.) FALLEN	$\frac{D}{T}$	RAW SC.	% SC.
1	A. K.	3.25	0	0.00	40	50
2	L. S.	3.25	4.78	1.59	46	70
3	R. S.	1.5	8.59	5.70	21	5
4	J. C.	3.25	.5	.17	50	80
5	C. O.	3.25	6.5	2.00	15	1
6	B. H.	3.25	0	0.00	11	1
7	P. E.	3.25	8.59	2.86	36	40
8	B. C.	3.25	0	0.00	33	30
9	L. L.	1.25	8.59	6.87	29	20
10	S. H.	3.25	0	0.00	42	60
11	M. B.	3.25	1.0	.33	64	95
12	L. P.	3.25	0	0.00	36	40
13	M. W.	2.5	8.59	3.44	57	90
14	J. S.	2.75	8.59	3.12	31	20
15	P. K.	3.25	0	0.00	30	20
16	S. B.	1.5	8.59	5.70	34	40
17	N. K.	3.25	3.18	1.06	23	5
18	H. H.	2.0	8.59	4.29	22	5
19	S. A.	.75	8.59	11.45	32	30
20	N. S.	3.25	1.11	.34	17	2
21	Y. R.	3.25	4.78	1.59	61	90
22	J. H.	3.25	0	0.00	43	60
23	M. S.	.75	8.59	11.45	23	5
24	B. D.	1.75	8.59	5.7	58	90
25	P. R.	3.25	3.81	1.27	26	10
26	B. S.	3.25	0	0.00	33	30
27	E. F.	.25	8.59	34.36	36	40
28	M. N.	3.00	8.59	2.86	50	80
29	A. D.	1.5	8.59	5.70	38	50
30	P. W.	3.25	0	0.00	30	20

TABLE 3
Correlations between Tests

Head-Falling and Pendulum.....	.06
Head-Falling and Adams-Lepley.....	-.10
Head-Falling and S-W.....	-.17
Head-Falling and Autokinetic.....	.05
Pendulum and Adams-Lepley.....	-.20
Autokinetic and Adams-Lepley.....	.04
S-W and Adams-Lepley.....	.33
S-W and Pendulum.....	-.19
S-W and Autokinetic.....	.25
Pendulum and Autokinetic.....	.56

With an N of 30, $sr = \frac{1}{\sqrt{N-1}} = .186$ (cf. p. 168).

gave no response to the suggestions presented on the record. It might be interesting to see if their response was negative but as our polygraph recorded only forward movement of the head, it would be impossible to tell from the graphs obtained whether a negative response was given in any of the trials. A sample graph of the movement of a subject is given (Fig. 2).

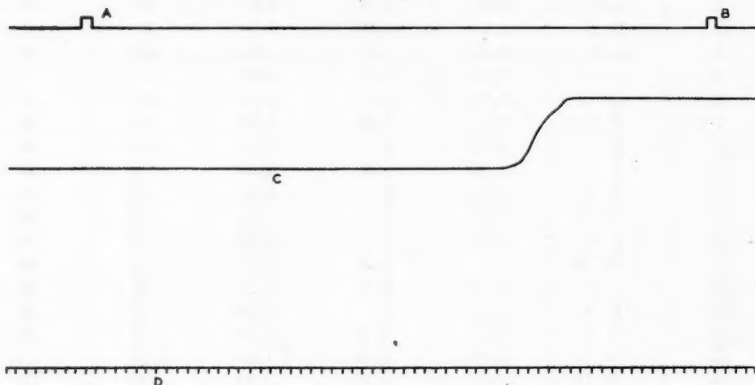


FIG. 2

A—B Onset and cessation of suggestion

C—Head Movement

D—Time in seconds

DISCUSSION

In the introduction to this paper, certain characteristics of suggestion were noted. Perhaps, however, it might be well to discuss briefly what prestige suggestion is. The difference between various acts brought about through suggestion and customary acts is that the former type is not intended. Ordinary acts take place because the individual wants to do them, but in the case of the former, the process is the response to suggestion and the acts are done, therefore, more unconsciously. Hull has set forth an hypothesis of voluntary action (14). He assumes that of habitual reactions, there are two levels—a symbolic and a non-symbolic (instrumental) level. Everyday acts such as chopping wood, cleaning streets, etc. he considers instrumental. Speech, however, in the ordinary sense of the word, is considered symbolic. The individual may exercise subvocal speech, thus stimulating proprioceptive sense organs which, in turn, send forth additional neural impulses. These may, then, produce instrumental acts. In the postural experiment, according to this system, or theory, the subject when told to relax and to think only of the suggestions given, is made more immune to his own symbolic activities. He is then more susceptible to the suggestions given externally. Subjects who do not respond readily to suggestion under these circumstances are thought to be less able to withdraw their symbolic activities to the extent that suggestion from an outside source may have any

effect. In the hypnotic state, which may be considered the ultimate of the suggestion states, these inner symbolic processes of the individual are at a minimum. And even after waking suggestion has been given, subjects are often drowsy which, according to Hull, is further evidence for believing that these symbolic activities have been lowered.

In the head-falling section of the experiment, the obtaining of a measure of susceptibility to suggestibility was being attempted. The introductory part of the phonograph record played suggested that the subject think of nothing except the words which were being spoken. The subject's mind was then "open" to the suggestion that his head was falling forward. The amount that this response was elicited gave a measure of the subject's susceptibility to suggestion. The higher the index score of the subject, the greater was his suggestibility. Due to the limitations of the polygraph used, negative suggestibility could not be measured. It was not the purpose of this experiment, however, to study this phenomenon. It is sufficient to say that Hull (14) found this to be existent in his experiments with postural sway but not to any great extent. These cases were rather uncommon.

Adams and Lepley by finding intercorrelations of the various sections of the Personal Audit have come to the conclusion that the tests involved do not measure the same characteristics, but rather each measures an independent factor (1). The reliability coefficient of the suggestibility test they found by the odd-versus-even method which was corrected by the Spearman-Brown prophecy formula. The coefficient of reliability for the suggestibility section of the Personal Audit was computed to be 0.91. A high score on this section is supposed to indicate suggestibility.

Chevreul's pendulum is an example of indirect heterosuggestion. In the instructions an indirect suggestion was made to the effect that the metal bar was magnetized. The degree to which the subjects responded by producing a swing in the bob was a measure of their suggestibility. Theoretically, the non-suggestible person holds the string and bob relatively stationary whereas the suggestible person hits the side of the container with the bob in a short time.

The phenomenon of autokinetic streaming³ was more or less an afterthought as far as this experiment was concerned. Ironically enough, the only valid correlation was found to exist between this illusion and Chevreul's pendulum. Voht has performed variations on the ordinary test of autokinetic streaming (28). His subjects were given twelve minutes to observe the phenomenon. The first two minutes were to be spent in a passive manner, observing the apparent movement. During the following two minutes, the subjects were instructed to attempt to facilitate the apparent movement of the light as much as possible. This was followed by four minutes of trying to inhibit the movement, and then

³ Relatively little work has been done recently on this although Guilford and Dallenbach (8) and later Guilford (9) made a detailed study of the autokinetic phenomenon by photographing the eye during the apparent movement of the light. They found that eye movements were not related to the apparent movement of the light and agreed with Ferree that perhaps autokinetic streaming is due to the streaming of the lymph over the retina.

two more minutes of passivity. He found that those with low indices facilitated poorly but could inhibit quite effectively. Those with indices in the middle range could both facilitate and inhibit well. The subjects having high indices could inhibit quite well but could not facilitate very effectively.

Although no experiment has been conducted to determine correlations between susceptibility to hypnosis and non-prestige suggestion, Aveling and Hargreaves have found relationships between non-prestige suggestion and two types of prestige suggestion (2). There was found to be a slight correlation between the prestige and non-prestige types of suggestion, but smaller relationships exist among the non-prestige tests. Brown has found certain relationships to exist among various tests of non-prestige suggestion (5). Hull states that this indicates a significant tendency, although this should not be relied upon too heavily due to the small groups used.

In the present experiment, no significant evidence was found to indicate that there is a positive relationship between the various tests of suggestibility employed in this experiment. In Table 3, the correlations are given and it may be seen that the correlation between the pendulum and autokinetic streaming is the only significant one (0.56). The relationship between the S-W and the Adams-Lepley, as shown by the co-efficients of correlation, is the only other one obtained which approaches a significant size. The application of Soper's (26) formula for the standard error of r ($\sigma_r = \frac{1}{\sqrt{N-1}}$) indicates that the chances of the pendulum-autokinetic correlation (0.56) being caused by chance are only 2/1000. In the case of the S-W and Adams-Lepley correlation (0.33) this drops to 5/100. There seems to be no definite explanation of why the pendulum-autokinetic correlation should be present. It may be that there is some factor of a personality nature or otherwise which is common to the autokinetic phenomenon and the pendulum and which is not common to the other tests given. Perhaps the structuration of the tests may have something to do with the relationship between them. The test of autokinetic streaming is the most unstructured of all of the tests given, in as much as there are no external limits placed on the subject's behavior. In this test, the field offers more opportunities for expression. The pendulum test is also fairly unstructured in so far as the oscillation of the bob is determined by forces operating within the subject herself. The other tests used are more structured. For example, in the head-falling test, the amount by which the head may fall is limited to 8.59cm in extent, and it can fall in only one direction. The Adams-Lepley is, perhaps, the most structured of all five tests as the subject is given only three choices in answering the question and must use one of these. The S-W illusion also limits the subject's behavior in that she is held within the boundaries of her judgment as to whether one of two external stimuli is stronger or weaker than the other. Behavioral deviation here is limited by the two externally-imposed stimuli. Perhaps these two tests between which there is a significant correlation permit more projection into the situation by the subject because of their relatively low structuration. It might be interesting to correlate the pendulum test with an introversion-extroversion test-such as the Bernreuter-to see if there is any relationship, as introverts have

been found to project themselves into the situation more than extroverts in the autokinetic streaming phenomenon (27). We might expect there to be, perhaps, a high correlation between the autokinetic and pendulum tests if a positive correlation were found between introversion and the amount of suggestibility shown in the pendulum test.

It was thought that perhaps a correlation of significant size might be found to exist between the S-W illusion and the other tests of suggestibility. This, however, was not found to be true. In our test, the greater the weight overestimate, the greater was the subject's susceptibility to the illusion. Due to the fact that this did not correlate with the other tests, a question arises as to whether this or any of the tests involved in the experiment are a measure in and by themselves of suggestibility. Perhaps they are, but in any event, the results were seen to be inconsistent. There does not seem to be, according to the evidence presented by this experiment, a generalized trait of suggestibility within individuals.

SUMMARY

This experiment was an attempt to study the relationship between the susceptibility to illusion and suggestibility. Thirty freshmen women were subjects and each was given five tests. These included the S-W illusion, autokinetic streaming, Chevreul's pendulum, the head-falling test and the suggestibility section of the Adams-Lepley Personal Audit. An attempt was made to see if there was any significant correlation among these tests. The only significant correlation found in this test was between Chevreul's pendulum and the phenomenon of autokinetic streaming. It seems difficult to explain this, although a possible explanation may lie in the fact that these two tests are the least structured of the five tests involved in this experiment. Also, it is possible that some personality factor is involved in these two tests which is not present in the other tests given. Perhaps this factor will be found in the introversion-extroversion continuum. Voth found that those having a higher index in the autokinetic test tended to be more introverted (27), but Porter has shown that extroverted women tend to be more suggestible (19). It is also true that no necessary relationship may exist between suggestibility and susceptibility to illusion.

CONCLUSIONS

1. It has been shown that of the tests used the only two possessing significantly common factors were Chevreul's pendulum and Voth's autokinetic. Precisely what factor or factors are common here must wait upon future research.
2. In so far as the tests used measure the behavior in question, no evidence was found for a general trait of suggestibility.
3. The clinical observations of Voth and Sexton are seriously in need of experimental check.
4. No precise or necessary relationship (except that between the pendulum and autokinetic tests) was found to exist between susceptibility to illusion and suggestibility.
5. This experiment itself needs further and more elaborate investigation.

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